Extension Theorems for Boundary Maps in Gromov Hyperbolic Spaces

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Abstract

We study the relationship of Gromov hyperbolic spaces to their boundary at infinity. We prove that to every rough isometry class of visual roughly geodesic Gromov hyperbolic space there corresponds a unique bilipschitz-quasimoebius class of complete quasimetric spaces and vice versa. Also, to every quasi-isometry class of visual roughly geodesic Gromov hyperbolic space there corresponds a unique power quasimoebius class of complete quasimetric spaces and vice versa.

We also prove that any bilipschitz-quasimoebius map between complete quasimetric spaces arises as the induced boundary map of a roughly isometric map between appropriate Gromov hyperbolic spaces. Similarly, every power-quasimoebius map between complete quasimetric spaces arises as the induced boundary map of a quasi-isometric map between appropriate Gromov hyperbolic spaces. In case the quasimetric space is moreover uniformly perfect, the quasi-isometric map is unique up to a bounded distortion.

Zusammenfassung

Wir untersuchen die Beziehungen von Gromov hyperbolischen metrischen Räumen zu ihren Rändern im Unendlichen. Wir zeigen, dass zu jeder groben Isometrieklasse von visuellen grob geodätischen Gromov hyperbolischen Räumen genau eine bilipschitz-quasimöbius Klasse von vollständigen quasimetrischen Räumen gehört und umgekehrt. Desweiteren gehört zu jeder Quasi-isometrieklasse von visuellen grob geodätischen Gromov hyperbolischen Räumen genau eine Power-quasimöbiusklasse von vollständigen quasimetrischen Räumen und umgekehrt.

Wir zeigen auch, dass jede bilipschitz-quasimöbius Abbildung zwischen vollständigen quasimetrischen Räumen als induzierte Randabbildung einer grob isometrischen Abbildung zwischen geeigneten Gromov hyperbolischen Räumen aufgefasst werden kann. Analog rührt jede power-quasimöbius Abbildung zwischen vollständigen quasimetrischen Räumen von einer quasi-isometrischen Abbildung zwischen geeigneten Gromov hyperbolischen Räumen her. Im Falle dass die quasimetrischen Räume uniform perfekt sind, ist diese quasi-isometrische Abbildung zudem eindeutig bestimmt bis auf eine beschränkte Störung.

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Chapter 1

Introduction and Outline

The starting point in the theory of Gromov hyperbolic spaces is an ingenious observation by Gromov that quadruples of points in the standard hyperbolic space $\mathbb{H}^n$ satisfy a condition, the so-called *δ-inequality*, which takes into account only properties of the metric. Consequently, this condition can be taken as a *definition* of hyperbolicity of arbitrary metric spaces. What is surprising is that this simple condition, while making absolutely no requirements on the space on any bounded scale, imposes strong conditions on the space on large scales that make it behave very similarly to negatively curved manifolds. Gromov hyperbolic spaces have been extensively studied over the last two decades, mostly with regard to geometric group theory. Some standard references include [Gro87], [BH99] Ch. III.H, [BS00], [BS07]. The reader mostly interested in the role Gromov hyperbolicity plays in geometric group theory can, for example, consult [Gro87], [GdlH90], [BM91], [Gro93], [Bow98b], [Bow98a] and references therein. The extension theorems we prove should be looked at as a "quasification" of theorems by Poincaré [Poi85] about the extension of Moebius maps on the boundary of classical hyperbolic space to Moebius maps (i.e. isometries) of the hyperbolic space itself. Previous results in this direction have been obtained by Paulin [Pau96], Bonk and Schramm [BS00], Buyalo and Schroeder [BS07], as well as Martínez-Peréz [Mar08]. The main results of this thesis have also been previously published in [Jor10].

After introducing some preliminary notation and remarks, we recall in Ch. 3 the basics of Gromov hyperbolic metric spaces. In Ch. 4 we discuss the Boundary at Infinity of such a space. We also discuss the notion of asymptotic curvature of a metric space and give an example of a phenomenon that so far has not appeared in the literature. In Ch. 5 we recall and generalize a construction of Buyalo and Schroeder to produce a Gromov hyperbolic space with prescribed Boundary at Infinity. Chapter 6 is the heart of this thesis and here we prove the fundamental Extension Theorems for boundary maps. These theorems unify and generalize the previous results cited above. We put our theorems in a functorial context in Ch. 7. In Ch. 8 we apply the Extension Theorems to deduce a very rigid relation between the linear asymptotic dimension of a visual roughly geodesic Gromov hyperbolic space and the Assouad-Nagata dimension of its boundary at infinity. Finally, in Ch. 9 we list some problems that remain open and that the author thinks would be worth exploring.

Chapter 2

Preliminaries

Here we gather some well known terminology and define some general terms that are so ubiquitous throughout the text that we find it appropriate to define them right away.

A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfies

1. $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)\ \forall x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$.

A metric space is called *proper* if closed and bounded sets are compact.

Remark 1. *We often find it convenient to use the notation $|xy|$, or even just xy for $d(x, y)$. It will be clear from the context that this refers to the distance between the points.*

The *Hausdorff distance* of two subsets $A, B \subset X$ of a metric space, denoted $d_H(A, B)$, is defined by

$$d_H(A, B) := \inf\{\epsilon > 0 \mid A \subset B_\epsilon \text{ and } B \subset A_\epsilon\},$$

where A_ϵ, B_ϵ are the *closed* ϵ-neighborhoods of A and B respectively, i.e. $A_\epsilon := \{x \in X \,|\mathrm{dist}(x, A) \leq \epsilon\}$.

If $A, B \subset X$ are subsets of a metric space, we say that A is a C*-net* in B, or A is C*-cobounded* in B, if $d_H(A, B) \leq C$. If we just say that A is a net in B, we mean that there exists a C such that A is a C-net in B.

Example 2. *$\mathbb{Z}$ is a $1/2$-net in $\mathbb{R}$.*

Definition 3. *A triple of real numbers $\{a, b, c\}$ is called a (additive) δ-*triple *if the two* smaller *numbers differ by at most δ. For example, if $a \leq b \leq c$, then we require $|a - b| \leq \delta$.*

The multiplicative version is

Definition 4. *A triple of real numbers $\{a, b, c\}$ is called a (multiplicative) K-*triple *if the two* larger *numbers have ratio at most K, i.e. $c/b \leq K$ if we again assume $a \leq b \leq c$.*

A metric space $(X,|\cdot|)$ that satisfies $|xz| \leq \max\{|xy|,|yz|\}$ for all $x,y,z \in X$ is commonly called an *ultrametric space*. Generalizing this inequality leads to *quasimetric spaces*.

Definition 5. *A* K-quasimetric space *is a set* Z *together with a map* $\rho : Z \times Z \to [0,\infty]$ *such that*

1. $\rho(z,y) \geq 0\ \forall z,y \in Z$*, with equality iff* $y = z$*,*
2. $\rho(z,y) = \rho(y,z)\ \forall z,y \in Z$*,*
3. $\rho(z,w) \leq K \max\{\rho(z,y), \rho(y,w)\}\ \forall w,y,z \in Z$*,*
4. *There is at most one* $z \in Z$ *such that* $\rho(z,y) = \infty$ *for all* $y \in Z \setminus \{z\}$*.*

If no point z *as in* 4 *exists,* Z *is said to be* non-extended*, while it is* extended *if there is such a* z *and this* z *is then called the* infinitely remote point*. By convention, a one-point space* $Z = \{z\}$ *is never extended.*

Property 3 above is equivalent to $\{\rho(x,y), \rho(x,z), \rho(y,z)\}$ being a multiplicative K-triple for any $x,y,z \in Z$.

A quasimetric ρ on a space Z induces a topology by declaring a set $A \subset Z$ to be open if for every $a \in A \setminus \{\infty\}$ there exists $r > 0$ such that $B_r^\rho(a) \subset A$, and if $\infty \in A$, then there exists $y \in Z$ and $r > 0$ such that $B_r(y)^c \subset A$. This topology is metrizable and in particular first-countable and Hausdorff. This follows from the fact that if (Z,ρ) is K-quasimetric, then (Z,ρ^s) is K^s-quasimetric (and the two topologies are clearly equivalent), and a result of Frink's ([Fri37]) whereby a K-quasimetric with $1 \leq K \leq 2$ is bilipschitz equivalent to a metric (extended if ρ is extended).

Here and in the future we always denote $B_r^\rho(x) := \{z \in Z | \rho(z,x) < r\}$ the *quantitatively* open balls, while $\overline{B}_r^\rho(x) := \{z \in Z | \rho(z,x) \leq r\}$ are the *quantitatively* closed balls. Note, though, that in contrast to the metric setting quantitatively open (closed) balls need not be topologically open (closed). For example, consider $Z := [0,1] \cup \{p\}$ with the 2-quasimetric ρ defined as the Euclidean distance for points on $[0,1]$, $\rho(p,t) := 1$ for $t \in [0,1/2]$ and $\rho(p,t) = 2$ for $t \in (1/2,1]$. Then $B_{3/2}(p) = [0,1/2] \cup \{p\}$ is not open.

Definition 6 (Completeness of a quasimetric)**.** *A quasimetric space* (Z,ρ) *is called* complete *if every Cauchy sequence in* Z *converges and if* ρ *is extended in case it is unbounded.*

We give some examples of quasimetric spaces.

Example 7.

1. *Every metric space is a* 2*-quasimetric space.*
2. *Every ultrametric space is a* 1*-quasimetric space.*
3. *The circle* S^1 *and* $\mathbb{R} \cup \{\infty\}$ *are complete quasimetric spaces, but* $\mathbb{R}$ *is not complete.*
4. *For the most important example, the boundary at infinity of a Gromov hyperbolic space, we refer to Section 4.2.1. They will turn out to be complete quasimetric spaces.*

The following definitions reflect the general philosophy of Gromov hyperbolic geometry, where bounded distortions of any kind are simply ignored.

Definition 8. *If a, b, C are real numbers, the notation $a \doteq_C b$ means $|a-b| \leq C$.*

We transfer the meaning to metric spaces by saying $x \doteq_C y$ if $|xy| \leq C$, where x, y are elements of some metric space X.

Similarly, for maps $F, G : X \to Y$ between metric spaces, $F \doteq_C G$ means $\sup_{x \in X} |F(x)G(x)| \leq C$. We say F and G are in the same rough mapping class *if $F \doteq_C G$ for some C.*

$G : Y \to X$ is called a rough inverse *of $F : X \to Y$ if $F \circ G \doteq_C \mathrm{id}_Y$ and $G \circ F \doteq_C \mathrm{id}_X$ for some C.*

Remark 9. *We usually omit the exact constant C because we do not care what exactly it is. Thus if we just write $F \doteq G$ it means that there is a C such that $F(x) \doteq_C G(x)$ for all x in the domains of F and G.*

Chapter 3

Gromov Hyperbolic Metric Spaces

3.1 Geodesic Spaces and Slim Triangles

Definition 10. *A* geodesic segment *in a metric space* $(X, |\cdot|)$ *is an isometric map* $\gamma : [a, b] \to X$*, where* $[a, b]$ *is a compact interval in* $\mathbb{R}$*.*

A geodesic ray *is an isometric map* $\gamma : [a, \infty) \to X$*, while an isometric map* $\gamma : (-\infty, \infty) \to X$ *is called a* bi-infinite ray*.*

Remark 11. *We frequently abbreviate and just speak of "a geodesic* $\gamma \ldots$*". It will always be clear from the context whether we mean a finite segment, a ray or a bi-infinite ray.*

Definition 12. *A metric space* X *is called* geodesic *if there exists a geodesic segment between any two of its points, i.e.* $\forall x, y \in X\ \exists a$ *geodesic* $\gamma : [a, b] \to X$ *with* $\gamma(a) = x, \gamma(b) = y$*.*

A space is uniquely geodesic *if for every choice of* x, y *there is a unique such geodesic* γ*.*

Remark 13. *We often abuse notation and write* xy *for a geodesic from* x *to* y*. If the space is not uniquely geodesic,* xy *stands for one arbitrary, but fixed, geodesic segment from* x *to* y*.*

A *triangle* in a metric space is the union $\gamma_0 \cup \gamma_1 \cup \gamma_2$ of three geodesics $\gamma_i, i = 0, 1, 2$, such that the endpoint of γ_i coincides with the starting point of $\gamma_{(i+1) \bmod 3}$. We also write xyz for a triangle with vertices x, y and z, cf. Rem. 13. To avoid technical difficulties of degenerate cases, we assume that the domains of all three geodesics have interior, i.e. the geodesics actually do form a triangle and not merely a line segment or a point.

A basic fact about the hyperbolic plane $\mathbb{H}^2$ is that any triangle is δ*-slim* in the sense that there is a constant $\delta = \delta(\mathbb{H}^2)$ such that for any triangle $\gamma_1 \cup \gamma_2 \cup \gamma_3$, γ_1 is contained in the δ-neighborhood of $\gamma_2 \cup \gamma_3$.

This leads us to the following very geometric definition of Gromov hyperbolicity, commonly attributed to Rips.

Definition 14. *A geodesic metric space is called* δ-hyperbolic *if for any triangle* $\gamma_1 \cup \gamma_2 \cup \gamma_3$*, one has that* γ_1 *is contained in the* δ*-neighborhood of* $\gamma_2 \cup \gamma_3$*.*

A geodesic metric space is called Gromov hyperbolic *if it is* δ*-hyperbolic for some* $\delta \geq 0$*.*

This definition has the disadvantage that, while very intuitive, it is not at all obvious how to extend it to spaces which are not necessarily geodesic. For this reason we choose another, but equivalent, approach.

Definition 15. *Suppose* xyz *is a triangle in an arbitrary metric space* X*. It is elementary to show that there exists a unique triple of points* $u \in xy, v \in yz, w \in zx$ *such that* $|xu| = |xw|$, $|yu| = |yv|$, $|zv| = |zw|$*. The points* u, v, w *are called the* equiradial points *of the triangle* xyz*.*

In fact, the points u, v, w *are determined by the quantities*

$$\begin{aligned} |xu| = |xw| &= \frac{1}{2}(|xy| + |xz| - |zy|), \\ |yu| = |yv| &= \frac{1}{2}(|yx| + |yz| - |xz|), \\ |zv| = |zw| &= \frac{1}{2}(|zy| + |zx| - |xy|). \end{aligned}$$

We make the following

Definition 16. *Let* X *a metric space,* $x, y, o \in X$*. The non-negative real number*

$$(x|y)_o := \frac{1}{2}(|ox| + |oy| - |xy|)$$

is called the Gromov product *of* x *and* y *w.r.t. the base point* o*.*

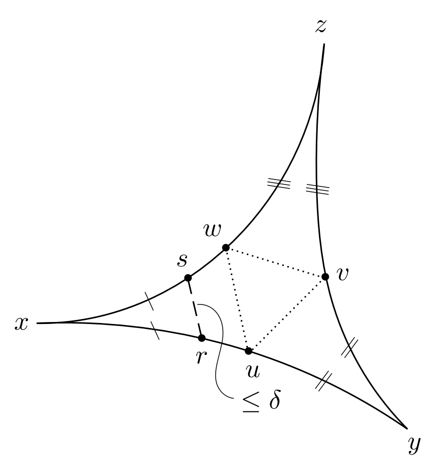

Figure 3.1: Equiradial points and thin triangles.

We have an elementary

Proposition 17. *Let* X *be a geodesic metric space that has the following property for some* $\delta \geq 0$*, cf. Fig. 3.1. Whenever* xyz *is a triangle in* X *and* $r \in xy$, $s \in xz$ *are points with* $|xr| = |xs| \leq (y|z)_x$*, then*

$$|rs| \leq \delta. \tag{A}$$

Then X is δ-hyperbolic in the sense of Def. 14. Conversely, any geodesic metric space that is δ_0-hyperbolic in the sense of Def. 14 satisfies condition (A) *with $\delta = 10\delta_0$. In particular, a geodesic metric space is Gromov hyperbolic if and only if it satisfies* (A) *for some $\delta \geq 0$.*

Proof. That a space which satisfies (A) for some δ is δ-hyperbolic in the sense of Def. 14 is obvious.

For the converse direction, note that the triangle inequality implies that the equiradial points u, v, w of xyz have pairwise distance no larger than $4\delta_0$. Let now $r \in xy, s \in xz$ with $|rx| = |sx| \leq (y|z)_x$ and consider the triangle xuw. Since $|uw| \leq 4\delta_0$, it follows from the triangle inequality that any $r \in xu$ with $|ru| \geq 5\delta_0$ must be δ_0-close to a point in wx and hence r must be $2\delta_0$-close to s. If $|ru| \leq 5\delta_0$ (and thus also $|ws| \leq 5\delta_0$), r may be δ_0-close to a point on uw. But then $|rs| \leq \delta_0 + 4\delta_0 + 5\delta_0 = 10\delta_0$. Hence X satisfies condition (A) with constant 10δ. $\square$

We give a few examples.

Example 18. *1. The standard hyperbolic spaces $\mathbb{H}^n$ of constant sectional curvature -1 are δ-hyperbolic for $\delta_{\mathbb{H}} = 2\ln\tau$, where τ is the solution of $t^2 = t + 1$.*

2. Every CAT(-1)*-space is $\delta_{\mathbb{H}}$ hyperbolic.*

3. A metric tree is 0*-hyperbolic. Conversely, every* 0*-hyperbolic geodesic metric space is a metric tree.*

We end this section with one of the most fundamental results about geodesic hyperbolic spaces; the stability of geodesics or, rather, of quasigeodesics. The analogous fact for classical hyperbolic space was originally proved by Morse, [Mor21] [Mor24]. In some sense, this theorem is really what makes the geometry of Gromov hyperbolic spaces accessible and allows to generalize many properties from classical hyperbolic space. For a proof, see for example [BH99] Thm. III.H.1.7.

Theorem 19 (Stability of geodesics)**.** *Let X be a geodesic δ-hyperbolic metric space and $\gamma : [0, a] \to X$ a (c, d)-quasigeodesic. There exists a constant $H = H(\delta, c, d)$ such that if η is any geodesic from $\gamma(0)$ to $\gamma(a)$, then* $\mathrm{im}(\gamma)$ *and* $\mathrm{im}(\eta)$ *are H-close in Hausdorff distance.* $\square$

Note that the constant H does *not* depend on a.

3.2 Hyperbolicity in General Metric Spaces

Property (A) from Prop. 17 leads to a definition of Gromov hyperbolicity in general metric spaces. The crucial point is the following δ-inequality, originally due to Gromov [Gro87].

Proposition 20 ([BS07] Prop. 2.1.2, 2.1.3)**.** *If a geodesic space X is δ-hyperbolic in the sense of property* (A)*, then*

$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta \tag{3.1}$$

for any $o, x, y, z \in X$.

Conversely, if a geodesic space X satisfies the δ-inequality (3.1) *for every $o, x, y, z \in X$, then it satisfies* (A) *with 4δ.* □

This allows us to give meaning to hyperbolicity in general metric spaces.

Definition 21 ([Gro87] 1.1)**.** *A metric space X is called* Gromov hyperbolic *if there is a δ such that every quadruple $o, x, y, z \in X$ of points in X satisify the δ-inequality* (3.1).

An equivalent formulation is as follows (recall Def. 3 for the notion of a δ-triple).

Proposition 22. *A metric space X is Gromov hyperbolic if and only if there is a δ such that for every quadruple of points in X, $o, x, y, z \in X$, the triple $\{(x|y)_o, (x|z)_o, (y|z)_o\}$ is a δ-triple.*

This is furthermore equivalent to the triple

$$\{-|xz| - |yo|, -|xo| - |yz|, -|xy| - |zo|\}$$

being a 2δ-triple for all $x, y, z, o \in X$.

Proof. Write out the definitions. □

Remark 23. *The triple $\{|xz| + |yo|, |xo| + |yz|, |xy| + |zo|\}$ is called the* cross-difference triple *of the quadruple (x, y, z, o).*

3.3 Morphisms Between Gromov Hyperbolic Spaces

We now come to one of the centerpieces of this work. Here we define the properties of the morphisms between hyperbolic spaces that we are going to look at. These are roughly isometric maps on the one hand and quasi-isometric maps or power quasi-isometric maps on the other hand. The first kind we can define without any further terminology.

Definition 24. *A map $F : X \to Y$ between metric spaces X and Y is called C-*roughly isometric *if $|F(x)F(y)| \doteq_C |xy|\ \forall x, y \in X$ (recall Def. 8).*

It is called roughly isometric *if it is C-roughly isometric for some C. It is called a* rough isometry *if there exists a D such that $F(X)$ is a D-net in Y.*

Definition 25. *A metric space X is called C-*roughly geodesic *if there exists for any $x, y \in X$ a C-*rough geodesic *joining x and y, where a C-rough geodesic is a C-roughly isometric map from an interval $I \subset \mathbb{R}$ into X.*

X is called roughly geodesic *if it is C-roughly geodesic for some C.*

To define an adequate setting for the definition of quasi- and power quasi-isometries, let us first introduce the notion of *cross-difference* of a quadruple of points.

Definition 26. *Let $Q = (x, y, z, w) \in X^4$ be an ordered quadruple in a metric space X. Define the* cross-difference *of Q, $cd(Q)$, as*

$$cd(Q) := (x|y)_o + (z|w)_o - (x|z)_o - (y|w)_o = \frac{1}{2}(|xz| + |yw| - |xy| - |zw|).$$

We will usually abuse notation and write $Q \subset X$ instead of $Q \in X^4$ for a quadruple in X, even though it is an *ordered* quadruple.

Definition 27. *Consider a map $F : X \to Y$ between metric spaces X and Y.*

*F is called (c, d)-*quasi-isometric *if*

$$\frac{1}{c}|xy| - d \leq |F(x)F(y)| \leq c|xy| + d \;\forall x, y \in X.$$

*F is called (c, d)-*power quasi-isometric *((c, d)-P-QI for short) if*

$$\frac{1}{c}cd(Q) - d \leq cd(F(Q)) \leq c \cdot cd(Q) + d \;\forall\; Q \in X^4.$$

F is called quasi-isometric *(*P-QI*) if it is (c, d)-quasi-isometric ((c, d)-P-QI) for some c, d.*

F is called a quasi-isometry *(*PQ-isometry*) if there exists a D such that $F(X)$ is a D-net in Y.*

Note that every P-QI map is also quasi-isometric (with the same constants). This follows from $cd(x, x, y, y) = |xy|$. Note also that every rough isometry, quasi-isometry and PQ-isometry has a *rough inverse*, cf. Def. 8, which is also a rough, quasi- or PQ-isometry respectively.

In the classical literature on hyperbolic spaces such as [Gro87], [BH99], [BS00], only quasi-isometric maps are considered. The notion of P-QI maps was introduced in [BS07]. The problem with quasi-isometric maps is that in general they do not preserve hyperbolicity, see example 28 2. below. This is why Buyalo-Schroeder introduced P-QI maps, which, by their simultaneous control of distances between *four* points instead of only two, are the appropriate class of morphisms between Gromov hyperbolic spaces as we will see below. A striking result of Buyalo-Schroeder (Thm. 31 below) says that a quasi-isometric map between *geodesic* hyperbolic spaces is in fact P-QI. Since in the classical literature, which was concerned mainly with applications to geometric group theory, only geodesic spaces were considered, there was no need for the more general concept of a P-QI map.

Example 28. *1. $F : \{10^n \,|\, n \in \mathbb{N}\} \to \mathbb{R}$, $F(n) := (-1)^n 10^n$ is quasi-isometric, but not P-QI. Both the domain and image are hyperbolic.*

2. If $F : \{(x, y) \,|\, y = |x|\} \to \mathbb{R}$ is the projection onto the x-axis, then F is quasi-isometric (even bilipschitz). The domain is not hyperbolic, as is easily seen. The image, however, is. F can thus not be P-QI (cf. Thm. 31 below). This example is attributed to Väisälä.

3. Consider the following space X, built from a basic building block T, a six-point space that is a subset of a tree, as in Fig. 3.2 (only the points A, B, C, D, E, F belong to X, the edges are for illustration only). X is obtained by taking a series of T_i, where within T_i we have the distances:

$$|A_iE_i| = |B_iE_i| = |D_iF_i| = |C_iF_i| := 10^i, \;|E_iF_i| := i.$$

Furthermore, define $|C_{i-1}A_i| := 10^{10^i}$, and finally let all other distances in X be defined as the length of the shortest path between the two vertices. Then of course X is hyperbolic, 0-hyperbolic in fact.

Consider now the map $F : X \to X$ that switches B_i and D_i, but leaves all other points of X fixed.

This F actually has a property that places it somewhere in between ordinary quasi-isometric maps and bona-fide P-QI maps, namely in contrast to general quasi-isometric maps F does *preserve the Gromov product in the sense that (the constants $1/2$ and 2 are not optimal)*

$$\frac{1}{2}(x|y)_o \leq (F(x)|F(y))_{F(o)} \leq 2(x|y)_o.$$

This can be easily verified because in trees the Gromov product $(x|y)_o$ is the length of the geodesic segment from o to the point where the geodesics ox and oy branch.

However, F is not *P-QI, because $cd(A_i, B_i, C_i, D_i) = 0$ while*

$$cd(A'_i, B'_i, C'_i, D'_i) = cd(A_i, D_i, C_i, B_i) = |E_iF_i| = i \to \infty.$$

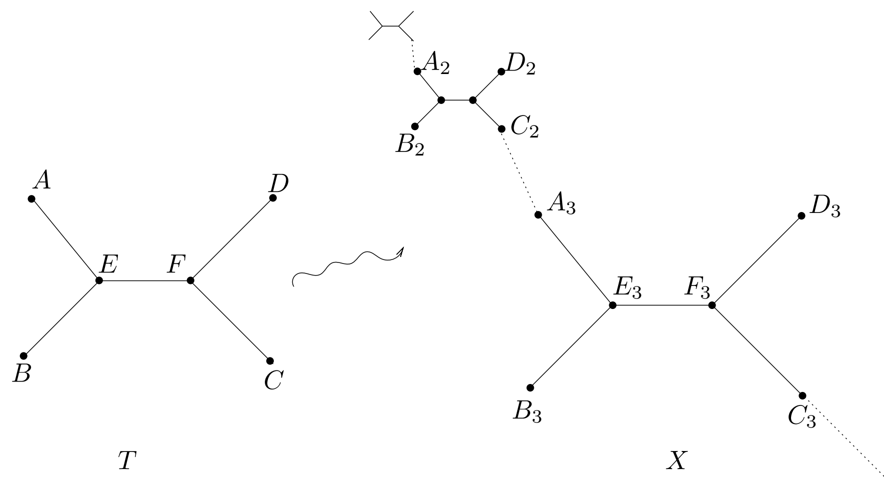

Figure 3.2: Building the space in Ex. 28 3.

Remark 29. *Example 28 3. is actually an example of a map which is power quasi-isometric as defined in [BS07] Def. 4.3.1. Our P-QI maps (Def. 27) are called* strongly power quasi-isometric *in [BS07] Def. 4.1.1. The difference is that P-QI maps as defined by Buyalo and Schroeder are maps which do preserve any* single *Gromov product in the usual "quasi-way", but they do not necessarily control the* difference of two Gromov products*, while our P-QI maps (or* strongly P-QI *in [BS07]) also preserve differences of Gromov products, which follows from the fact that $\frac{1}{2}(|xz| + |yw| - |xy| - |zw|) = (x|y)_z - (w|y)_z$. See also [BS07] Lemma 4.2.3 and Prop. 4.3.2.*

Proposition 30. *If $F : X \to Y$ is P-QI then X is Gromov hyperbolic if and only if $F(X) \subset Y$ is Gromov hyperbolic.*

Proof. $\Rightarrow$: Suppose the cross-difference triple $\{r, s, t\}$ of a quadruple of points in X is a δ-triple and F is (c, d)-P-QI. Assume w.l.o.g. $r \leq s \leq t$ and denote by

r', s', t' the appropriate quantities w.r.t. the images under F. If $r' \leq s' \leq t'$ or $r' \leq t' \leq s'$ it is immediate that $\{r', s', t'\}$ is a $(c\delta + d)$-triple. It only remains to check the case $t' \leq r' < s'$. But since $0 \leq r' - t' \leq c(r-t) + d \leq d$, $\{r', s', t'\}$ is a d-triple in this case.

$\Leftarrow$: Follows from the fact that a rough inverse of a P-QI map is also P-QI. $\square$

To finish the section, we cite the mentioned theorem of Buyalo-Schroeder which says that it is not necessary to distinguish between quasi-isometric and P-QI maps in the setting of geodesic spaces. The proof is based on the stability of geodesics, Thm. 19.

Theorem 31 ([BS07] Thm. 4.4.1)**.** *If X, Y are geodesic Gromov hyperbolic metric spaces with hyperbolicity constants δ and δ' respectively and if $F : X \to Y$ is a (c, b)-quasi-isometric map. Then F is (c, d)-P-QI, where $d = d(c, b, \delta, \delta')$.* $\square$

Example 32 (Gromov hyperbolic groups)**.** *Consider a finitely generated group with a symmetric generating set $G =< S|R >$. Associated to this presentation is the Cayley graph Γ of G, whose vertices are the group elements and two vertices v, v' are joined by an edge exactly when $v = v's_i$ for some generator s_i. Define the length of every edge to be 1 and consider the induced length metric on Γ. Then Γ is clearly a geodesic metric space.*

If S' is a different (symmetric) generating set for G, then the word-lenghts l w.r.t S and l' w.r.t. S' are quasi-isometric to each other, which is easily seen by considering the longest and shortest elements of one generating set in the word-length of the other generating set. By Thm. 31, the Cayley graph Γ associated to S is thus a geodesic Gromov hyperbolic metric space if and only if the Cayley graph associated to Γ' is.

A group is called a Gromov hyperbolic group *if the Cayley graph associated to one (and hence any) generating set is a Gromov hyperbolic space.*

Gromov hyperbolic groups have played an important role in geometric group theory, geometric topology and algorithmics in recent years, see the references we listed in the Introduction.

Chapter 4

Boundary at Infinity of a Gromov Hyperbolic Space

To every Gromov hyperbolic metric space X one can associate a so-called *boundary at infinity*, $\partial_\infty X$, of X. This is a (quasi)metric space which encodes to a certain degree what the space X looks like on large scales. The idea and the construction of the space is in some sense analogous to the Tits boundary of Hadamard manifolds or CAT(0) spaces.

4.1 The Boundary as a Set

4.1.1 The Geodesic Boundary $\partial_g X$

To underscore the similarities to the theory of Tits boundaries in CAT(0)-spaces we first introduce the *geodesic boundary at infinity.* Even though this set will in general only be a subset of the "real" boundary at infinity defined in the next section, it turns out that for *proper* geodesic Gromov hyperbolic spaces the two definitions are equivalent.

Definition 33 (Asymptotic rays)**.** *Let X be a Gromov hyperbolic metric space and $\gamma, \gamma' : [0,\infty) \to X$ two geodesic rays. We say γ is* asymptotic to γ', $\gamma \sim \gamma'$, *if $\sup_t |\gamma(t)\gamma'(t)| < \infty$, or what is the same (triangle inequality!), $d_H(\gamma, \gamma') < \infty$.*

The relation to be asymptotic is obviously an equivalence relation and we define.

Definition 34. *The* geodesic boundary at infinity *of a Gromov hyperbolic space X, $\partial_g X$, is the set of equivalence classes of asymptotic rays in X.*

Remark 35. *Clearly if a Gromov hyperbolic space X does not have a lot of geodesics, the geodesic boundary will typically be very small. For example, the Gromov hyperbolic space $\mathbb{Z}$ has no geodesic rays, thus $\partial_g X = \emptyset$. But even for geodesic spaces the geodesic rays in general do not capture the whole asymptotic geometry of X. We give an example in the next section.*

4.1.2 The Gromov Boundary $\partial_\infty X$

In the general construction of $\partial_\infty X$ the rays used for $\partial_g X$ are replaced with *sequences converging to infinity.*

Definition 36. *A sequence $\{x_i\} \subset X$ in a Gromov hyperbolic metric space X is said to* converge to infinity *if $(x_i|x_j)_o \to \infty$ for some $o \in X$.*

Note that $|(x|y)_o - (x|y)_{o'}| \leq |oo'|$, so the definition does not depend on the base point o.

Definition 37 (Asymptotic sequences)**.** *Two sequences $\{x_i\}, \{y_i\}$ in X are called* asymptotic*, or* equivalent*, if $(x_i|y_i)_o \to \infty$ for some, and hence any, $o \in X$.*

Because $\{(x_i|y_i)_o, (x_i|y_i)_o, (y_i|z_i)_o\}$ is a δ-triple for each i, the relation to be asymptotic is an equivalence relation among sequences converging to infinity.

Definition 38. *The* Gromov boundary at infinity*, or just* boundary at infinity*, $\partial_\infty X$, of a Gromov hyperbolic metric space X is the set of equivalence classes of sequences converging to infinity.*

Elements of $\partial_\infty X$ are usually denoted by lower case greek letters $\xi, \eta, \zeta, \ldots$.

We obviously have $\partial_g X \subset \partial_\infty X$. By an Arzelà-Ascoli argument, one can show that if X is proper and geodesic, then to every $\xi \in \partial_\infty X$ and every $o \in X$ there exists a ray γ from o to ξ. For non-proper spaces this argument does not work and in general $\partial_g X \subsetneq \partial_\infty X$.

Example 39. *Consider the following metric graph X, cf. Fig. 4.1. Take the non-negative real half-line $\mathbb{R}_{\geq 0}$. Add for each $k \geq 1$ an edge of length 1 from 0 to $1 + 1/2^k$. From each of the endpoints $1 + 1/2^k$ of these edges, draw one edge of length 1 to $2 + 1/2^{k-1}$ for $k \geq 2$. Continue like this by drawing an edge of length 1 from $n + 1/2^k$ to $(n+1) + 1/2^{k-1}$ for each $n \geq 2$ and $k \geq 2$.*

X is a geodesic Gromov hyperbolic space, but it is impossible to define an infinite geodesic. In particular, $\partial_g X = \emptyset$. But clearly, $\partial_\infty X = \{\omega\}$ is a one-point set.

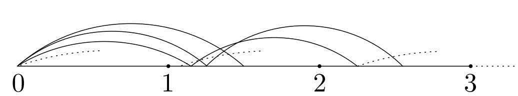

Figure 4.1: Geodesic hyperbolic space with no infinite geodesics.

4.2 The Boundary as a Quasimetric Space

We now introduce quasimetrics on the set $\partial_\infty X$.

4.2.1 Gromov Product on the Boundary

We have seen that if X is a δ-hyperbolic space, then

$$\{(x|y)_o, (x|z)_o, (y|z)_o\}$$

is a δ-triple for all $x, y, z, o \in X$. We now extend the Gromov product to $\partial_\infty X$ as follows. Let $\xi, \xi' \in \partial_\infty X$ and $o \in X$. Set

$$(\xi|\xi')_o := \inf \liminf_{i\to\infty}(x_i|x_i')_o,$$

where the infimum is taken over all sequences $(x_i) \in \xi$, $(x_i') \in \xi'$. It is a fact ([BS07] Lemma 2.2.2(2)) that with this definition the δ-inequality extends to the boundary at infinity. That is,

$$\{(\xi|\xi')_o, (\xi|\xi'')_o, (\xi'|\xi'')_o\}$$

is a δ-triple for all $\xi, \xi', \xi'' \in \partial_\infty X$. Now example 7, 3 becomes clear. If X is a δ-hyperbolic space, $a > 1$, $o \in X$ and $(\cdot|\cdot)_o$ denotes the Gromov product with respect to the base point o, then $a^{-(\cdot|\cdot)_o}$ is an a^δ-quasimetric on the set $\partial_\infty X$. Note that this quasimetric is always bounded by 1, because the Gromov product is greater or equal to 0, $(\cdot|\cdot)_o \geq 0$.

4.2.2 Busemann Functions and Inversions

In the previous section we have seen how to put a *bounded* quasimetric on $\partial_\infty X$. Now in the classical setting for the hyperbolic plane $\mathbb{H}^2$, the boundary comes in two different shapes, once as S^1, the boundary of the unit disk model, and once as $\mathbb{R}\cup\{\infty\}$, the boundary of the upper half plane model. The two spaces, S^1 and $\mathbb{R}\cup\{\infty\}$ are related via the stereographic projection, a Moebius map. The quasimetrics $a^{-(\cdot|\cdot)_o}$ we introduced in §4.2.1 are the analogs of S^1. Our goal in this section is to introduce a second type of quasimetrics on the boundary which should play the role that $\mathbb{R}\cup\{\infty\}$ does in the classical case.

The crucial point is to realize that stereographic projection $S^1 \to \mathbb{R}\cup\{\infty\}$ is in fact an example of an inversion in a circle, namely the circle that is centered at the north pole $(0, 1)$ and has radius $\sqrt{2}$, cf. [BS07] §5.3.2 for details. And there is a formula for how the distance changes when one applies such an inversion. For example, invert $\mathbb{R}^2 \cup \{\infty\}$ in the unit circle. Denote the inversion by ι. Then if $(r_1, \phi_1), (r_2, \phi_2)$ are polar coordinates of two points we get

$$\begin{aligned} ||\iota(r_1,\phi_1) - \iota(r_2,\phi_2)|| &= ||(1/r_1,\phi_1) - (1/r_2,\phi_2)|| \\ &= \sqrt{1/r_1^2 + 1/r_2^2 - 2/(r_1r_2)\cos(\phi_1-\phi_2)}, \end{aligned}$$

while

$$||(r_1,\phi_1) - (r_2,\phi_2)|| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\phi_1-\phi_2)}.$$

In other words,

$$||\iota(r_1,\phi_1)-\iota(r_2,\phi_2)|| = \frac{||(r_1,\phi_1)-(r_2,\phi_2)||}{||(r_1,\phi_1)||\cdot||(r_2,\phi_2)||}.$$

In general, one can show (cf. [BS07] §5.3.2) that if the inversion circle is centered at the point (r_o,ϕ_o) and has radius r, then

$$||\iota(r_1,\phi_1)-\iota(r_2,\phi_2)|| = \frac{r^2||(r_1,\phi_1)-(r_2,\phi_2)||}{||(r_1,\phi_1)-(r_o,\phi_o)||\cdot||(r_2,\phi_2)-(r_o,\phi_o)||}.$$

This leads us to define the following.

Definition 40 (Inverted quasimetric)**.** *If (Z,ρ) is a quasimetric space and $o\in Z\setminus\{\infty\}$, $r>0$, the* inversion ρ' of ρ at center o with radius r *is defined by*

$$\rho'(u,v) := \frac{r^2\rho(u,v)}{\rho(u,o)\rho(v,o)}.$$

It is not difficult to show (see [BS07], proof of Prop 5.3.6) that ρ' is a K^2-quasimetric whenever ρ is a K-quasimetric.

If now X is some Gromov hyperbolic space X, $o\in X$ and $\omega\in\partial_\infty X$, define the function

$$\begin{aligned} b_{o,\omega} : X &\to \mathbb{R} \\ x &\mapsto |ox| - 2(\omega|x)_o, \end{aligned}$$

and consider the following *Gromov product based at ω*:

$$(x|y)_{b_{o,\omega}} := \frac{1}{2}(b_{o,\omega}(x)+b_{o,\omega}(y)-|xy|).$$

A trivial computation shows $(x|y)_{b_{o,\omega}} = (x|y)_o - (\omega|x)_o - (\omega|y)_o$. This suggests that $a^{-(\cdot|\cdot)_{b_{o,\omega}}}$ is the inversion of $a^{-(\cdot|\cdot)_o}$ with inversion center ω and radius 1. Of course, for this statement to make sense we first have to extend $(\cdot|\cdot)_{b_{o,\omega}}$ to $\partial_\infty X$. This is done just as in the case of $(\cdot|\cdot)_o$, namely

$$(\xi|\xi')_{b_{o,\omega}} := \inf\liminf_{i\to\infty}(x_i|x'_i)_{b_{o,\omega}},$$

where the infimum is taken over all sequences $(x_i)\in\xi, (x'_i)\in\xi'$.

The function $b_{o,\omega}$ is an example of a *Busemann function*, which are well-known in the theory of non-positively curved manifolds. The complete set of Busemann functions for a Gromov hyperbolic space is defined as follows.

Definition 41 (Cf. e.g. [BS07] Def. 3.1.3)**.** *Let X be a δ-hyperbolic space and $\omega\in\partial_\infty X$ fixed. The set $\mathcal{B}(\omega)$ of all* Busemann functions based at ω *consists of all those functions $b : X\to\mathbb{R}$ for which there exists $o\in X$ and a constant $c\in\mathbb{R}$ such that $b \doteq_{2\delta} b_{\omega,o}+c$.*

The δ-inequality carries over to these Gromov products as well.

Proposition 42 ([BS07] Lemma 3.2.4(2))**.** *For X a δ-hyperbolic space and $\xi,\eta,\zeta,\omega\in\partial_\infty X$ arbitrary, the numbers $(\xi|\eta)_b$, $(\xi|\zeta)_b$, $(\eta|\zeta)_b$ form a 22δ-triple for any $b\in\mathcal{B}(\omega)$.* □

This shows that the quasimetric $a^{-(\cdot|\cdot)_{b_{o,\omega}}}$ on $\partial_\infty X$ is bilipschitz equivalent to the quasimetric obtained by inverting $a^{-(\cdot|\cdot)_o}$ in ω and with radius 1.

We are now in a position to define the most general form of the boundary at infinity.

Definition 43. *Let X be a Gromov hyperbolic space, $a > 1$, $o \in X$, $b \in \mathcal{B}(\omega)$ for some $\omega \in \partial_\infty X$.*

The symbol $\partial_\infty^{a,o} X$ denotes the quasimetric space $(\partial_\infty X, a^{-(\cdot|\cdot)_o})$.

The symbol $\partial_\infty^{a,b}$ denotes the quasimetric space $(\partial_\infty X, a^{-(\cdot|\cdot)_b})$.

It is well-known that the boundary at infinitiy of a Gromov hyperbolic space is a complete quasimetric space. For a proof, see e.g. [BS00] Prop. 6.2 (the proof carries over verbatim to the quasimetric setting).

4.3 Quasimoebius and Quasisymmetric Maps

In Section 3.3 we described roughly isometric and (power)-quasi-isometric maps between Gromov hyperbolic spaces. The Extension Theorems in Chapter 6 will put these maps in relation to two classes of maps between the boundaries of the given hyperbolic spaces. In a rather explicit sense, the properties of maps between hyperbolic spaces are "exponentiated" to yield the appropriate properties for boundary maps. Roughly isometric maps will be related to *bilipschitz* and *bilipschitz-quasimoebius* maps and quasi-isometric maps will be related to so-called *power quasisymmetric* and *power quasimoebius* maps. These quasimoebius maps are characterized by how they control the *cross-ratio* of a quadruple of points.

Definition 44. *Let (Z, ρ) be a quasimetric space. For any quadruple $Q = (x, y, z, w) \in Z^4 \setminus D$, where $D \subset Z^4$ is the subset where the same point appears three or four times, define the* cross-ratio *of Q, $cr(Q)$, by*

$$cr(Q) = \frac{\rho(x, z)\rho(y, w)}{\rho(x, y)\rho(z, w)}.$$

Remark 45. *If a point appears more than once in a quadruple Q, we define $cr(Q)$ via the following conventions (where distinct letters denote distinct points and ω denotes the infinitely remote point of Z, if it exists)*

$$\begin{aligned} cr(x, x, y, z) &:= \infty \qquad cr(x, y, x, z) := 0 \\ cr(x, x, x, y) &:= 1 \\ cr(x, y, z, \omega) &:= \frac{\rho(x, z)}{\rho(x, y)} \end{aligned}$$

$$cr(x, y, \omega, \omega) = cr(x, x, \omega, \omega) := \infty \qquad cr(x, \omega, y, \omega) = cr(x, \omega, x, \omega) := 0$$

Definition 46. *If $\theta : [0, \infty) \to [0, \infty)$ is a homeomorphism, an injective map $f : Z \to Z'$ between quasimetric spaces is called θ-quasimoebius (θ-QM) if*

$$\frac{1}{\theta(\frac{1}{cr(Q)})} \leq cr(f(Q)) \leq \theta(cr(Q)).$$

f is called power quasimoebius *(P-QM) if it is θ-QM for a θ of the form $\theta(t) = q \max\{t^{1/p}, t^p\}$. It is called* bilipschitz quasimoebius *(BL-QM) if θ can be taken linear, $\theta(t) = \lambda t$.*

Closely related to QM maps are *quasisymmetric* (QS) maps, which are the injective maps that preserve the ordinary ratio sr of a triple (x, y, z), $sr(x, y, z) := \rho(x, z)/\rho(x, y)$, in an analogous way.

Note that a non-constant quasisymmetric map is automatically injective. The same is true for quasimoebius maps.

We refer to [Väi85] and [BS07] Ch. 5, for more information on quasimoebius and quasisymmetric maps. We just note that every quasisymmetric map is quasimoebius and that quasimoebius maps are homeomorphisms onto their images. In particular, they map complete spaces to complete spaces.

Remark 47. *In fact, the bilipschitz class of $\partial_\infty^{a,o}X$ does not depend on $o \in X$ and the quasimoebius class depends on neither of the parameters. Thus we may suppress one or both of them and just write $\partial_\infty^a X$, or $\partial_\infty X$. Whenever we do this it is to be understood that the statement holds for any admissible choice of the omitted parameter(s).*

4.4 Induced Maps Between Boundaries

If a map $F : X \to X'$ between Gromov hyperbolic spaces maps sequences going to infinity in X to sequences going to infinity in X' and equivalent sequences to equivalent sequences, then F induces a map between boundaries, which we denote $\partial_\infty F : \partial_\infty X \to \partial_\infty X'$.

For example, every roughly isometric map $F : X \to X'$ induces an injection $\partial_\infty F : \partial_\infty X \to \partial_\infty X'$. A quasi-isometric map $F : X \to X'$ between *geodesic* hyperbolic spaces induces a boundary map by the stability of geodesics (cf. [BH99] Thm. III.H.1.7). However, the map $F : \{10^i | i \in \mathbb{N}\} \to \mathbb{R}$, $F(10^i) := (-1)^i 10^i$ is quasi-isometric, but does not induce a boundary map in any reasonable sense. This is another reason why quasi-isometric maps are in general not the right maps to look at in the setting of hyperbolic metric spaces (we have already seen that quasi-isometric maps need not preserve hyperbolicity). In contrast, any P-QI map *does* induce a map between associated boundaries. This follows from $cd(x, y, o, o) = (x|y)_o$.

Remark 48. *In fact, also the map of Ex. 28 3 induces a canonical boundary map, because it preserves Gromov products in a quasi-way. It is not P-QI according to our definition, however. The reason we avoid maps of this type and consider only P-QI maps is that it seems to be impossible to recover non-P-QI maps from the boundary maps they induce, much in contrast to P-QI maps, cf. Thm. 101.*

We quote now well-known results about maps between Gromov hyperbolic spaces that induce maps between the boundaries associated to the spaces. Recall that omitting values for base points or Busemann functions in $\partial_\infty^{a,o}, \partial_\infty^{a,b}$ means that the statement is valid for any choice of them.

Theorem 49 (Cf. e.g. [BS07] Thm. 5.2.10)**.** *If $F : X \to X'$ is a roughly isometric map between Gromov hyperbolic spaces, then for each $a > 1$ F induces a bilipschitz-quasimoebius map $\partial_\infty F : \partial_\infty^a X \to \partial_\infty^a X'$.*

If $F : X \to X'$ is a power quasi-isometric map between Gromov hyperbolic spaces, then F induces a power quasimoebius map between associated boundaries, $\partial_\infty F : \partial_\infty^a X \to \partial_\infty^{a'} X'$, where $a, a' > 1$. □

4.5 Asymptotic Curvature and Visual Metrics

This section is independent of the major results of this thesis and skipping it will have no ill effects on the understanding of the rest of this work. Our purpose in this section is to give an example of a visual geodesic Gromov hyperbolic space with asymptotic curvature -1 such that $e^{-(\cdot|\cdot)_o}$ is *not* bilipschitz equivalent to any metric on $\partial_\infty X$.

As mentioned in the previous section, for every Gromov hyperbolic space X and $o \in X$, $a^{-(\cdot|\cdot)_o}$ becomes bilipschitz equivalent to an honest metric on $\partial_\infty X$ when a is close enough to 1. The notion of *asymptotic curvature* of a Gromov hyperbolic space was introduced by Bonk and Foertsch in [BF06] and, at least for visual spaces, it quantifies just how close to 1 a has to be taken. It is defined as follows.

Definition 50. *Let X be a metric space and $\kappa \in [-\infty, 0)$. We say that X has an* asymptotic curvature bound κ*, or X is* $AC_u(\kappa)$*, if there exists $p \in X$ and a constant $c \geq 0$ such that for all $z, z' \in X$ and all chains $z = x_0, x_1, \ldots, x_n = z'$ in X we have*

$$(z|z')_p \geq \min_i (x_{i-1}|x_i)_p - \frac{1}{\sqrt{-\kappa}} \log n - c,$$

where $\frac{1}{\sqrt{\infty}} := 0$.

The asymptotic curvature of X*, $K_u(X)$, is then given by*

$$K_u(X) := \inf\{\kappa \mid X \textit{ is an } AC_u(\kappa)\textit{-space}\}.$$

If we parametrize a by e^ϵ with $\epsilon > 0$, we have the following connection between the asymptotic curvature of X and visual metrics on $\partial_\infty X$.

Theorem 51 ([BF06] Thm. 1.5)**.** *Let X be a visual Gromov hyperbolic metric space. Then*

$$K_u(X) = -b^2,$$

where

$$b := \sup\{\epsilon > 0 \mid \textit{there exists a visual metric on } \partial_\infty X \textit{ with parameter } \epsilon\}.\square$$

There do not appear any examples in the literature of visual Gromov hyperbolic spaces where the supremum b is not attained. This is why we construct in this section one such example, namely a visual Gromov hyperbolic space X with $K_u(X) = -1$ but such that $e^{-(\cdot|\cdot)_o}$ is *not* bilipschitz to any metric on $\partial_\infty X$.

The example actually uses theory from later chapters, namely hyperbolic approximation and extension theorems, but since this section is not used later in this thesis, it causes no problems to place it here. We first describe the boundary at infinity that we want X to have, and then define X as a hyperbolic approximation of this boundary. The crucial point is that the boundary is a quasimetric space but the chain construction of Frink's does not produce a metric but merely a pseudo-metric. But if $\partial_\infty X$ were bilipschitz to a metric d, then the chain construction would of course yield a metric d' that is bilipschitz to d.

The example is a modification of an example by Schroeder [Sch06] which was designed to show the limits of Frink's chain construction. The following paragraphs up to Prop. 52 are quoted verbatim from [Sch06], with a small tweak when defining the lengths of edges in the graph.

The boundary is constructed as follows. Let Z be the set of dyadic rational of the interval $[0,1]$. Then Z is the disjoint union of Z_n , $n \in \mathbb{N}$, where $Z_0 = \{0,1\}$, and $Z_n = \{\frac{k}{2^n} : 0 < k < 2^n, k\ odd\}$ for $n \geq 1$. If $z \in Z_n$, we say that the level of z is n and write $\ell(z) = n$. For the following construction it is useful to see Z embedded by $z \mapsto (z, \ell(z))$ as a discrete subset of the plane. Let $z = \frac{k}{2^n} \in Z_n$ with $n \geq 1$, then we define the right and the left neighbors $l(z) = \frac{k-1}{2^n}$ and $r(z) = \frac{k+1}{2^n}$. We see that $\ell(l(z)), \ell(r(z)) < n$ and clearly $l(z) < z < r(z)$, where we take the usual ordering induced by the reals. Given $z \in Z$ with $\ell(z) \geq 1$ we consider the *right path* $z, r(z), r^2(z), \ldots$ and the left path $z, l(z), l^2(z), \ldots$. Note that after a finite number of steps the right path always ends at 1 and the left path always ends at 0.

We use the following facts:

Fact 1: Consider for an arbitrary $z \in Z$ the levels of the vertices on the right and on the left path, i.e. $\ell(l(z), \ell(l^2(z)), \ldots$ and $\ell(r(z), \ell(r^2(z)), \ldots$. Then all intermediate levels n with $0 < n < \ell(z)$ occur exactly once (either on the right or on the left path). E.g. consider $11/64$ which is of level 6. The left path is $11/64, 5/32, 1/8, 0$ (containig the intermediate levels 5 and 3), the right path is $11/64, 3/16, 1/4, 1/2, 1$ (containing the remaining intermediate levels $4, 2$ and 1). This fact can be verified by looking to the dyadic expansion of z, e.g. $11/64 = 0.001011$. Note that the dyadic expression of $l(z)$ is obtained from the one of z by removing the last 1 in this expression, i.e. $l(0.001011) = 0.00101$. The dyadic expression of $r(z)$ is obtained by removing the last consequetive sequence of 1's and putting a 1 instead of the 0 in the last entry before the sequence, e.g $r(0.001011) = 0.0011$. Therefore the levels of the left path (resp. of the right path) correspond to the places with a 1 (resp. with a 0) in the dyadic expansion.

Fact 2: Let $l^k(z)$ be a point on the left path and $\ell(l^k(z)) \geq 1$. Let m be the integer, such that $r^m(z)$ is the first point on the right path with $\ell(r^m(z)) < \ell(l^k(z))$, then $r(l^k(z)) = r^m(z)$. A corresponding statements holds for points on the right path. This fact can also be verified by looking to the dyadic expension.

We consider the graph whose vertex set is Z, and the edges are given by the pairs $\{0,1\}, \{z, r(z)\}, \{z, l(z)\}$, where the $z \in Z$ are points with level ≥ 1. One can visualize this graph nicely, if we use the realization of Z in the plane described above. In this picture we can see the edges as line intervalls and the graph is planar. In this picture the left path from a point z with $\ell(z) \geq 1$ can be viewed as the graph of a piecewise linear function defined on the interval $[0, z]$ (here $z \in [0,1]$) and the the right path as the graph of a piecewise linear function on $[z, 1]$. The union of these two paths form a "tent" in this picture.

Fact 3: Below this tent lies no point of Z.

To every edge in this graph we associate a length. To the edge $\{0,1\}$ we associate the length 1, and to an edge of the type $\{z, l(z)\}$ and $\{z, r(z)\}$ we associate the length $1/(\ell(z)2^{\ell(z)})$. Now we define the quasimetric ρ. First set $\rho(0,1) = 1$. Let $z, z' \in Z$ be points such that z, z' is not the pair $0, 1$. Let us assume $z < z'$. Then we consider the right path $z, r(z), r^2(z), \ldots, 1$ starting from z, and the left path $z', l(z'), l^2(z'), \ldots, 0$ starting from z'. Then the properties from above imply that these two paths intersect at a unique point $r^k(z) = l^s(z')$. Then we obtain a V-shaped path $z, r(z), \ldots, r^k(z) = l^s(z'), \ldots, l(z'), z'$ formed

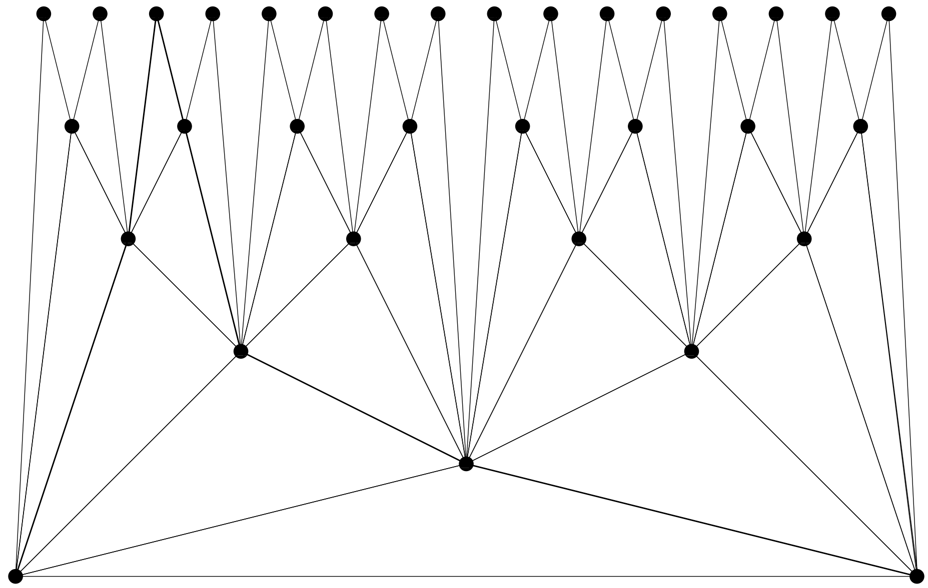

Figure 4.2: Graph with a tent.

by edges from our graph from z to z'. We define $\rho(z, z')$ to be the sum of the lengths of the edges of this path.

Proposition 52. *ρ is a K-quasimetric on Z with $K = 8$.*

Proof. The proof is analogous to the one in §2 of [Sch06]. First note that for any $z \in Z_n$,

$$\tau_n := \rho(0, z_n) + \rho(z_n, 1) = \sum_{k=1}^{n-1} \frac{1}{2^k k} + \frac{2}{2^n}.$$

It follows that $1 = \tau_1 > \tau_2 > \cdots > \tau_\infty$, where τ_∞ is the limit of τ_n. Obviously $\tau_\infty \geq 1/2$.

Consider now the following special triangle z_0, z_1, z_2, with the properties, that:

z_1 lies on the left path starting from z_0,

z_2 lies on the right path starting from z_0,

z_2 lies on the right path starting from z_1.

These conditions imply that $z_1 \leq z_0 \leq z_2$ and $\ell(z_0) \geq \ell(z_1) \geq \ell(z_2)$.

Now

$$\rho(z_1, z_0) + \rho(z_0, z_2) - \rho(z_1, z_2) = \tau_n - \tau_m < 0. \tag{4.1}$$

Consequently, $\rho(z_1, z_2) \geq \rho(z_0, z_1) + \rho(z_0, z_2)$. We now show that it is not much larger.

We want a uniform C such that

$$\rho(z_1, z_2) \leq C(\rho(z_0, z_2) + \rho(z_0, z_2)). \tag{4.2}$$

Now $\rho(z_1, z_2) - (\rho(z_0, z_2) + \rho(z_0, z_2)) = \tau_m - \tau_n$, i.e.

$$1 - \frac{\rho(z_0, z_2) + \rho(z_0, z_2)}{\rho(z_1, z_2)} = \frac{\tau_m - \tau_n}{\rho(z_1, z_2)},$$

or

$$\rho(z_1, z_2) = \Big(1 - \frac{\tau_m - \tau_n}{\rho(z_1, z_2)}\Big)^{-1}(\rho(z_1, z_0) + \rho(z_0, z_2)).$$

To prove (4.2) it thus suffices to find a uniform C such that

$$\frac{\tau_m - \tau_n}{\rho(z_1, z_2)} \leq 1 - 1/C,$$

which is equivalent to (with a different but uniform C)

$$\rho(z_1, z_2) \geq (1 + 1/C)(\tau_m - \tau_n). \tag{4.3}$$

Now $\tau_m - \tau_n = \frac{1}{2^m m} - \frac{1}{2^{m+1}(m+1)} - \ldots - \frac{1}{2^{n-1}(n-1)} - \frac{2}{2^n n}$. Since $\rho(z_1, z_2)$ is the sum of the lengths of edges between the level m of z_1 and the level of z_2, we trivially have $\rho(z_1, z_2) \geq 1/(2^m m)$. Therefore (4.3) follows if we find a uniform C such that

$$\frac{1}{2^m m} \geq (1 + \frac{1}{C})\Big(\frac{1}{2^m m} - \frac{1}{2^{m+1}(m+1)} - \ldots - \frac{1}{2^{n-1}(n-1)} - \frac{2}{2^n n}),$$

or even more so if

$$\frac{1}{C}\frac{1}{2^m m} \leq \frac{1}{2^{m+1}(m+1)} + \ldots + \frac{1}{2^{n-1}(n-1)} + \frac{2}{2^n n}.$$

Trivially, $C = 4$ will do the job.

Remains to prove $\rho(z_1, z_2) \leq C(\rho(z_0, z_1) + \rho(z_0, z_2))$ for three arbitrary points z_0, z_1, z_2.

We may assume that the vertices are such that $z_1 < z_0 < z_2$, since by (4.4) the side $z_1 z_2$ is the longest side in the triangle.

Consider the V-shaped path from z_1 to z_2, let $\tilde{z} = r^k(z_1) = l^s(z_2)$ be the "lowest" point on this path. By symmetry of the whole argument we assume without loss of generality that $z_0 \leq \tilde{z}$. Now (using Fact 3) we see that the left path staring at z_0 will intersect the right path starting in z_1. Let z_1' be the intersection point. Let z_2' be the first point, where the right path starting at z_0 coincides with the right path starting at z_1.

To begin with, we have

$$\rho(z_1, z_2) \geq \rho(z_1, z_0) + \rho(z_0, z_2) \tag{4.4}$$

because $\rho(z_1', z_2') \geq \rho(z_1', z_0) + \rho(z_0, z_2')$.

On the other hand, we have

$$\frac{\rho(z_1, z_0) + \rho(z_0, z_2)}{\rho(z_1, z_2)} \geq \frac{\rho(z_1', z_0) + \rho(z_0, z_2')}{\rho(z_1', z_2')}, \tag{4.5}$$

for

$$\frac{\rho(z_1', z_1) + \rho(z_1', z_0) + \rho(z_0, z_2') + \rho(z_2', z_2)}{\rho(z_1, z_1') + \rho(z_1', z_2') + \rho(z_2', z_2)} - \frac{\rho(z_1', z_0) + \rho(z_0, z_2')}{\rho(z_1', z_2')} \geq 0$$

if and only if

$$\begin{aligned}\rho(z_1', z_2')(\rho(z_1', z_1) + \rho(z_1', z_0) + \rho(z_0, z_2') + \rho(z_2', z_2)) - \rho(z_1, z_1')(\rho(z_1', z_0) + \rho(z_0, z_2'))\\ - \rho(z_1', z_2')(\rho(z_1', z_0) + \rho(z_0, z_2')) - \rho(z_2', z_2)(\rho(z_1', z_0) + \rho(z_0, z_2')) \geq 0\end{aligned}$$

if and only if

$$\rho(z_1', z_2')(\rho(z_1', z_1) + \rho(z_2', z_2)) - \rho(z_1, z_1')(\rho(z_1', z_0) + \rho(z_0, z_2') \\ - \rho(z_2', z_2)(\rho(z_1', z_0) + \rho(z_0, z_2')) \geq 0$$

if and only if

$$\rho(z_1, z_1') \underbrace{(\rho(z_1', z_2') - \rho(z_1', z_0) - \rho(z_0, z_2'))}_{\geq 0} + \rho(z_2', z_2) \underbrace{(\rho(z_1', z_2') - \rho(z_1', z_0) - \rho(z_0, z_2'))}_{\geq 0} \geq 0$$

which is indeed the case, since the sums in the parentheses are both ≥ 0 by (4.1). This shows that (4.5) holds. By (4.2) (where $C = 4$), we get

$$\rho(z_1, z_2) \leq 4(\rho(z_0, z_1) + \rho(z_0, z_2)) \; \forall z_0,\, z_1,\, z_2 \in Z,$$

which together with (4.4) proves that ρ is an 8-quasimetric on Z. □

Proposition 53. *The chain construction on (Z, ρ) does not produce a metric on Z, let alone one which is bilipschitz to ρ.*

Proof. It suffices to exhibit a sequence $(\sigma_n)_n$ of chains between 0 and 1 such that their sums tend to 0. So consider $\sigma_n = (0, 1/2^n, 2/2^n, \ldots, (2^n - 1)/2^n, 1)$. Then $\sum(\sigma_n) = 2^n \cdot 1/n2^n = 1/n$. □

Definition 54. *Let γ be a V-shaped path whose vertices we denote in increasing order by $z_0 < z_1 < \ldots < z_n$. A point $z \in Z$ is said to be* above *the path γ if both the left and right path of z intersect γ. Note that this in particular implies $z_0 \leq z \leq z_n$.*

A point is said to be below *γ if it is not above γ.*

A path η, that is a sequence of neighboring vertices, is said to be above or below the path γ if each vertex of η is above or below γ respectively.

A chain $\sigma = (z_0, \ldots, z_n)$ is said to be above or below γ if each v_i^{i+1} is above or below γ respectively.

Lemma 55. *If $x < y$ then the right path of y lies above the right path of x. On the other hand, the left path of y lies below the left path of x.*

Proof. As both right paths of x and y end in 1 they certainly intersect, so it is enough to show that the left paths of $y, r(y), r^2(y), \ldots, r^n(y) = 1$ intersect the right path of x. But clearly $r^k(y) > y$ for every k, so this always happens as we already saw in the definition of the quasimetric ρ, cf. p. 30. □

Definition 56. *For a given quasimetric space (Z, ρ), if $\sigma = (z_0, \ldots, z_n)$ is a chain in Z and $\alpha > 0$ we denote by $\sum_\alpha(\sigma)$ the* length *of the chain with respect to ρ^α, i.e.*

$$\sum_{i=1}^{n}{}_\alpha(\sigma) := \rho^\alpha(z_{i-1}, z_i).$$

The next lemma says that chains between two points with any chance to infimize the length of chains must be monotone and strictly between the endpoints, i.e. $z_0 < z_1 < \cdots < z_n$.

Lemma 57 (Monotonicity of Chains)**.** *Let $x < y \in Z$ arbitrary and $\sigma = (x = z_0, z_1, \ldots, z_n = y)$ an arbitrary chain from x to y.*

Suppose there exists an index i such that $z_i > z_n$. Let j be the first subsequent index such that $z_j < z_n$, or $j = n$ if there is no such z_j. Let $\hat{\sigma} = (z_0, z_{i-1}, z_j, \ldots, z_n)$ be the chain obtained by throwing away the z_k which are larger than z_n. Then $\sum_\alpha(\hat{\sigma}) \leq \sum_\alpha(\sigma)\ \forall \alpha \in (0, 1]$.

Suppose that $z_k < z_n$ for all $k = 0, \ldots, n-1$ and that there exists an index i such that $z_i > z_{i+1}$. Let $j > i$ be the first subsequent index for which $z_j > z_i$. Let $\hat{\sigma} = (z_0, \ldots, z_i, z_j, \ldots, z_n)$ be the chain where the backtracking part of σ has been spliced out. Then $\sum_\alpha(\hat{\sigma}) \leq \sum_\alpha(\sigma)\ \forall \alpha \in (0, 1]$.

Proof. Assume first $z_j > z_n$ for all $n > j > i$. Then

$$\begin{aligned}\sum_\alpha(\sigma) - \sum_\alpha(\hat{\sigma}) &= \rho^\alpha(z_0, z_1) + \ldots + \rho^\alpha(z_{n-1}, z_n)\\ &\quad -\Big(\rho^\alpha(z_0, z_1) + \ldots + \rho^\alpha(z_{i-2}, z_{i-1}) + \rho^\alpha(z_{i-1}, z_n)\Big)\\ &\geq \rho^\alpha(z_{i-1}, z_i) - \rho^\alpha(z_{i-1}, z_n).\end{aligned}$$

Now, since $z_n < z_i$, the left path of z_n lies above the left path of z_i by Lemma 55. In particular, if the former intersects the right path of z_{i-1} at a point x and the latter at a point y, then $\ell(x) \geq \ell(y)$. If $\ell(x) > \ell(y)$, then $\rho(z_{i-1}, z_i) - \rho(z_{i-1}, z_n) \geq \frac{1}{\ell(y)2^{\ell(y)}} - \sum_{s=\ell(x)}^{\ell(z_n)} \frac{1}{s2^s} > 0$. If $\ell(x) = \ell(y)$ then between x and z_n on the left path of z_n there is a vertex q where the left paths of z_n and z_i merge. Since $z_i > z_n$, the length of the edge with lower vertex q on the left path of z_n has length smaller than $1/2$ the length of the edge with lower vertex q on the left path of z_i. Since the portion of the paths between q and x agree, this shows $\rho(z_{i-1}, z_i) > \rho(z_{i-1}, z_n)$ also in this case.

If j is the first index $> i-1$ with $z_j < z_n$, we claim $\rho^\alpha(z_{i-1}, z_j) \leq \rho^\alpha(z_{i-1}, z_i)$. This follows for the same reason as above (i.e. consider the right path of z_{i-1} and the left paths of z_i and z_j). But then certainly $\sum_\alpha(\hat{\sigma}) \leq \sum_\alpha(\sigma)$.

Consider now the second case with the indices as described. It is enough to show $\rho^\alpha(z_{i-1}, z_j) < \rho^\alpha(z_{i-1}, z_i)$, but this follows immediately as above from $z_{i-1} < z_j < z_n < z_i$. □

The next lemma allows us to restrict our attention to chains which lie "above" the v-shaped path between its endpoints.

Lemma 58. *Let $x < y \in Z$. Then any monotone chain from x to y lies above V_x^y.*

Proof. Suppose first that y lies on the right path of x. $z_0 = x$ lies above v_x^y. It is immediate that V_0^1 lies above V_x^y, for it is a V-shaped path from $z_0 = x$ to a point $z_1 > x$. Next, V_1^2 lies above the right path of z_1 by the same argument, and the right path of z_1 lies above that of x by Lemma 55 and the claim follows.

The same argument also works if $x < y$ arbitrary, i.e. y not necessarily on the right path of x. For if z is the lowest point of V_x^y then V_x^y is the concatenation of V_x^z and V_z^y. If a chain σ contains z_i, z_{i+1} such that V_i^{i+1} dipped below V_x^y, then it cannot be monotone as there would be a z_j such that V_{j-1}^j reaches z_j via its right neighbor, i.e. V_{j-1}^j contains the edge $\{z_j, r(z_j)\}$, which implies $z_j < z_{j-1}$. Thus a monotone chain lies above V_x^y. □

Lemma 59. *Let $x < y$ and z the lowest point on V_x^y, the V-shaped path from x to y. Let σ be a chain such that each V_{i-1}^i, the V-shaped path between z_{i-1} and z_i, lies above V_x^y and $x < z_1 < z_2 < \ldots < z_{n-1} < y$. Then σ passes through z, i.e. there exists an index i such that $z \in V_i^{i+1}$.*

Proof. Let i such that $z_i \leq z < z_{i+1}$. It is enough to show that the right path of z_i passes through z, for by a symmetric argument it will follow that the left path of z_{i+1} also passes through z.

But it was already discussed in the construction of ρ, cf. p. 30, that if $z_i < z$ then the right path of z_i intersects the left path of z in a unique point. Since the left path of z is below V_x^y (with z being the lowest point on that path), and the right path of z_i stays above the right path of x, the right path of z_i must thus pass through z itself. □

The next lemma says that on large levels, (Z, ρ) looks asymptotically like a 2-quasimetric space.

Lemma 60. *Consider the sequence $K_m = 2(1 + 1/m)$. For each m there is a level N_m such that if $x, y, z \in Z$ are three points such that every vertex on the triangle V_x^y, V_x^z, V_z^y has level at least N_m, then*

$$\rho(x, y) \leq K_m \max\{\rho(x, z), \rho(z, y)\}.$$

Proof. Consider first a special triangle as in the proof of Prop. 52. Then if $\ell(y) =: n > m := \ell(x)$,

$$\begin{aligned}\rho(x, z) - \rho(x, y) - \rho(z, y) &= \rho(0, x) + \rho(x, 1) - \rho(0, y) - \rho(y, 1) \\ &= \sum_{k+1}^{m-1} \frac{1}{k2^k} + \frac{2}{m2^m} - \sum_{k+1}^{n-1} \frac{1}{k2^k} - \frac{2}{n2^n} \\ &= \frac{1}{m2^m} - \frac{1}{(m+1)2^{m+1}} - \ldots - \frac{1}{(n-1)2^{n-1}} - \frac{2}{n2^n}\end{aligned} \tag{4.6}$$

The last quantity is > 0, but if $K_m := 2(1 + 1/m)$, then

$$\rho(x, z) \leq K_m(\rho(x, y) + \rho(y, z)),$$

for $K_m \frac{1}{(m+1)2^{m+1}} = \frac{2}{m2^{m+1}} = \frac{1}{m2^m}$, and (4.6) becomes negative.

Now if $x < y < z$ is an arbitrary triangle as in the statement of the lemma then we get just as in the proof of Prop. 52 that the same inequalities with K_m hold. □

Corollary 61. *For any given $\alpha \in (0, 1)$ there is an integer N_α such that if $x, y, z \in Z$ are three points such that every vertex on the triangle V_x^y, V_x^z, V_z^y has level at least N_α, then we have*

$$\rho^\alpha(x, y) \leq 2 \max\{\rho^\alpha(x, z), \rho^\alpha(z, y)\}.$$

Proof. Choose an integer m such that $K_m^\alpha \leq 2$ and set $N_\alpha = m$.

$$\rho(x, z) \leq K_m(\rho(x, y) + \rho(y, z))$$

implies

$$\rho^\alpha(x,z) \le K_m^\alpha(\rho(x,y)+\rho(y,z))^\alpha \le 2(\rho^\alpha(x,y)+\rho^\alpha(y,z)).$$

□

Corollary 62. *Let $\alpha \in (0,1)$ fixed. Suppose $x < y$, and V_x^y lies entirely above level N_α. There exists a constant C_α such that for any chain σ from x to y we have*

$$\sum{}_\alpha(\sigma) \ge C_\alpha \rho^\alpha(x,y),$$

where $\sum_\alpha$ denotes the length of the chain σ in (Z,ρ^α), $\sum{}_\alpha(\sigma) := \sum \rho^\alpha(z_{i-1},z_i)$.

Proof. Consider the subset of $S \subset Z$ consisting of all vertices that lie above V_x^y. By Lemmas 57, 58 it is enough to consider chains σ that are monotone and lie above V_x^y. Such chains lie in particular in S. Then $(S,\rho^\alpha|_S)$ is a K-quasimetric space with $K \le 2$ by Corollary 61, and so the chain construction will produce a metric that is bilipschitz to ρ^α. □

Lemma 63. *Let $x < y \in Z$. For any $\alpha \in (0,1)$ there is a C_α such that $\sum_\alpha(\sigma) \ge C_\alpha \rho^\alpha(x,y)$ for any chain σ from x to y.*

Proof. Again, by Lemmas 57, 58 it is enough to consider chains σ that are monotone and lie above V_x^y. We also may assume $\ell(y) < N_\alpha$, otherwise the claim follows from Corollary 62. Now by monotonicity, it is evident that σ has no more than M_α of its z_i below level N_α (there are *in total* only finitely many elements of Z below level N_α), where M_α is a natural number only depending on N_α (thus only on α).

If $\sigma = (z_0,\dots,z_n)$, denote by $i_1,\dots i_k$, $k \le M_\alpha$ the indices where V_i^{i+1} crosses the level N_α. Then $i_j + 1 \le i_{j+1}$, equality may occur. Now, with the understanding that $\sum{}_\alpha(\sigma|_{i_j+1,\dots,i_{j+1}}) = 0$ in case $i_j+1 = i_{j+1}$, we have

$$\begin{aligned}
\sum{}_\alpha(\sigma) &= \sum{}_\alpha(\sigma|_{0,\dots,i_1}) + \rho^\alpha(z_{i_1},z_{i_1+1}) + \sum{}_\alpha(\sigma|_{i_1+1,\dots,i_2}) + \rho^\alpha(z_{i_2},z_{i_2+1}) + \cdots \\
&\ge \frac{1}{C_\alpha}\rho^\alpha(z_0,z_{i_1}) + \rho^\alpha(z_{i_1},z_{i_1+1}) + \frac{1}{C_\alpha}\rho^\alpha(z_{i_1+1},z_{i_2}) + \rho^\alpha(z_{i_2},z_{i_2+1}) + \cdots \\
&\quad \cdots + \frac{1}{C_\alpha}\rho^\alpha(z_{i_k+1},z_n) \\
&\ge \frac{1}{C_\alpha}\rho^\alpha(z_0,z_n),
\end{aligned}$$

where the C_α changes from line to line but always remains a uniform constant only depending (ultimately) on α. For parts of the chain completely above N_α, use C_α from Lemma 62, for parts of it completely under N_α, use the fact that it has to be a finite chain (in fact, a uniform bound on the number of elements exists) in an 8-qm space (each (Z,ρ^α) is in particular 8-quasimetric) and for the last line again use that the chain which we are left with on the penultimate line is a chain with uniform bound on the number of elements (bounded by M_α). □

This last lemma effectively proves the following proposition.

Proposition 64. *For any $\alpha \in (0,1)$, the chain construction on (Z,ρ^α) produces a metric which is bilipschitz to ρ^α.* □

Now denote by $X := \mathrm{Hyp}_{1/e}(Z,\rho)$ the hyperbolic approximation with parameter $1/e$ of (Z,ρ).

Proposition 65. *X is a visual geodesic Gromov hyperbolic space with asymptotic upper curvature $AC_u(X) = -1$. However, $\partial_\infty^{e;o} X$ is not bilipschitz to a metric space.*

Proof. It is well-known that Z injects the set $\partial_\infty X$ and the quasimetric $e^{-(\cdot|\cdot)_o}$ restricted to Z is bilipschitz to ρ, say with constant C. Thus if $e^{-(\cdot|\cdot)_o}$ were bilipschitz to a metric then the same would hold for ρ, which contradicts Prop. 53.

Remains to show that $e^{-\alpha(\cdot|\cdot)_o}$ is bilipschitz to a metric on $\partial_\infty X$. Since $e^{-(\cdot|\cdot)_o}$ is bilipschitz to ρ on Z, $e^{-\alpha(\cdot|\cdot)_o}$ is bilipschitz to ρ^α on Z. Now we use that Z lies dense in $\partial_\infty X$ (w.r.t. any of the topologies induced by $e^{-(\cdot|\cdot)_o}$, $e^{-\alpha(\cdot|\cdot)_o}$). In fact it is known that $\partial_\infty X$ is the completion of Z.

So let $\sigma = (x = z_0, \ldots, z_n = y)$ any chain in $\partial_\infty X$. Suppose $e^{-(\cdot|\cdot)_o}$ is a K-quasimetric. Now by density of Z in $\partial_\infty X$ we clearly can find points $\tilde{z}_i \in Z$ such that for every i, $e^{-(z_i|\tilde{z}_i)_o}$ is small enough so that $e^{-(\tilde{z}_{i-1}|\tilde{z}_i)_o} \leq K^2 e^{-(z_{i-1}|z_i)_o}$. The same inequality also holds for $e^{-\alpha(\cdot|\cdot)_o}$. Thus

$$
\begin{aligned}
\sum{}_\alpha(\sigma) := \sum e^{-\alpha(z_{i-1}|z_i)_o} &\geq \frac{1}{K^2}\sum e^{-\alpha(\tilde{z}_{i-1}|\tilde{z}_i)_o} \\
&\geq \frac{1}{CK^2}\sum \rho^\alpha(\tilde{z}_{i-1},\tilde{z}_i) \\
&\geq \frac{1}{C_\alpha C K^2}\rho^\alpha(\tilde{z}_0,\tilde{z}_n) \\
&\geq \frac{1}{C_\alpha C^2 K^2} e^{-\alpha(\tilde{z}_0|\tilde{z}_n)_o} \\
&\geq \frac{1}{C_\alpha C^2 K^4} e^{-\alpha(x|y)_o},
\end{aligned}
$$

where the last inequality also requires that $\tilde{z}_0$ and $\tilde{z}_n$ are much closer to z_0 and z_n respectively than to each other, but they can clearly be chosen so that this is satisfied. $\square$

Chapter 5

Hyperbolic Approximation

The idea of a hyperbolic approximation is to construct for a given metric space Z a Gromov hyperbolic space X such that $\partial_\infty X = Z$, in some suitable sense.

This procedure appears in the literature through various sorts of cone construction on Z, which are a sort of a warped products analogon adapted to the "rough" setting of Gromov hyperbolic spaces. Classical sources for this approach include [TV99] and [BS00]. In [BS07] Buyalo and Schroeder further developed constructions of Elek [Ele97], and Bourdon and Pajot [BP03] to give a very intuitive geometric construction of such a space X. Not only is their approach to the construction very elementary and illuminating, it also produces a particularly nice space X, namely a metric graph. Probably just because of the elementary nature of this construction, we obtain quite a clear view on the structure of X, and we use this knowledge in various ways to prove the extension theorems and the uniqueness of extensions in Chapter 6.

Moreover, since the operation of taking a boundary at infinity of a Gromov-hyperbolic space canonically gives rise to *quasi-metrics* on the boundary (rather than honest metrics) we would like to be able to perform hyperbolic approximation directly on such a quasi-metric space, instead of first introducing a non-canonical visual metric and *then* approximating this latter metric space. It is another feature of Buyalo and Schroeder's construction that it translates readily to the setting of quasimetric spaces.

5.1 The Construction

Let (Z, ρ) be a complete K-quasimetric space. Let $r \leq 1/K^3$. The procedure now goes as follows. For every $k \in \mathbb{Z}$ let V_k be a maximal r^k-separated subset of Z (such exist by Zorn), where r^k-separated means $\rho(v, v') \geq r^k$ for all $v, v' \in V_k$. Denote by $\mathcal{V}$ the set of all ordered pairs (k, z) with $k \in \mathbb{Z}$ and $z \in V_k$. The projection $\ell : \mathcal{V} \to \mathbb{Z}$ to the first coordinate is called *level function*, and $\ell(v)$ the *level* of v, while the projection $\pi : \mathcal{V} \to Z$ to the second coordinate sends v to its *center* $\pi(v) \in Z$.

Remark 66. *Sometimes the notation $\pi(v)$ becomes too cumbersome so that we often identify a point $v \in V_k$ with its center $\pi(v) \in Z$. The notation $\rho(v, w)$ is thus interpreted to mean $\rho(\pi(v), \pi(w))$.*

Remark 67 (Hereditary vertex systems)**.** *Also by a Zorn-type argument there exist* hereditary *vertex systems $\mathcal{V} = \{V_k\}_k$, meaning that $\pi(V_k) \subset \pi(V_{k+1})$. Working with such hereditary systems often simplifies arguments and we will use them without reservation when it suits us, namely in the proofs of the Extension Theorems in Ch. 6.*

The *hyperbolic approximation of Z with parameter $r \leq 1/K^3$*, denoted $\mathrm{Hyp}_r(Z, \rho)$ or $\mathrm{Hyp}_r(Z)$ for short, is now defined to be the simplicial graph with vertex set $\mathcal{V}$, where two vertices $v, w \in \mathcal{V}$ are joined by an edge exactly when

- $\ell(v) = \ell(w)$ and the sets $B(v) := B_{Kr^{l(v)}}(\pi(v))$ and $B(w) := B_{Kr^{l(w)}}(\pi(w))$ intersect in Z, or
- $\ell(v) = \ell(w) + 1$ and $B(v)$ is contained in $B(w)$.

The first point is slightly different from the definition in [BS07], §6.1. We opt to use open balls for technical reasons.

5.2 Metric Structure of $\mathrm{Hyp}\, X$

For Z a *metric* space, Buyalo and Schroeder proved that $\mathrm{Hyp}_r(Z)$ is a hyperbolic space with all of the desired properties (i.e. Thm. 74 holds). Even though the proofs are easily adapted to the quasimetric setting, we here include, for the sake of completeness, the rewritten proofs of the lemmata in [BS07], §§6.2, 6.3 which lead up to the desired Theorem 74.

Lemma 68 ([BS07] Lemma 6.2.1)**.** *For every $v \in V$ there is a vertex $w \in V$ with $\ell(w) = \ell(v) - 1$ radially connected to any horizontal neighbor of v.*

Proof. Let $\ell(v) = k + 1$ and $w \in V_k$ such that $\rho(v, w) < r^k$ and v' a horizontal neighbor of v. Then $z \in B(v')$ means $\rho(z, v') < Kr^{k+1}$. Let $s \in B(v) \cap B(v')$.

$$\begin{aligned}
\rho(z, w) &\leq K \max\{\rho(w, v), \rho(v, z)\} \\
&\leq K \max\Big\{\rho(w, v), K \max\{\rho(v, v'), \rho(v', z)\}\Big\} \\
&\leq K \max\left\{\rho(w, v), K \max\Big\{K \max\{\rho(v, s), \rho(s, v')\}, \rho(v', z)\Big\}\right\},
\end{aligned}$$

which, since $\rho(v, s), \rho(s, v') < Kr^{k+1}$, implies $\rho(z, w) < Kr^k$, where we used that $K^4 r^{k+1} \leq Kr^k$, i.e. $r \leq 1/K^3$.

□

Lemma 69 ([BS07] Lemma 6.2.2)**.** *For every $v, v' \in \mathcal{V}$ there exists $w \in \mathcal{V}$ with $\ell(w) \leq \ell(v), \ell(v')$ such that v, v' can be connected to w by radial geodesics. In particular, the space X is geodesic.*

Proof. Let $\ell(v) = k$ and $\ell(v') = k'$. Choose $m < \min\{k, k'\}$ small enough such that $\rho(v, v') \leq r^{m+1}$. Applying Lemma 68, we find radial geodesics $\gamma = v_k v_{k-1} \dots v_m$ and $\gamma' = v'_{k'} v'_{k'-1} \dots v_{m'}$ in X connecting $v = v_k$ and $v' = v_{k'}$

respectively with the m-th level. It follows from the definition of radial edges that $v \in B(u)$, $v' \in B(u')$ for every vertex $u \in \gamma, u' \in \gamma'$. Then

$$\begin{aligned}\rho(v', v_m) &\leq K \max\{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v')\} \\ &\leq K \max\{\rho(v_m, v_{m+1}, K \max\{\rho(v_{m+1}, v), \rho(v, v')\}\},\end{aligned}$$

hence $\rho(v_m, v') < Kr^m$. □

Lemma 70 ([BS07] Lemma 6.2.3)**.** *Assume that $|vv'| \leq 1$ are horizontal neighbors. Then any w, w' radially connected to v and v' respectively are horizontal neighbors if $\ell(w) = \ell(w')$.*

Proof. $B(v) \cap B(v') \neq \emptyset$ and $B(v) \subset B(w), B(v') \subset B(w')$ imply $B(w) \cap B(w') \neq \emptyset$. □

Corollary 71 ([BS07] Corollary 6.2.4)**.** *For any two radial geodesics γ, γ' with common ends, the distance between vertices of common levels is at most 1.* □

The rest of [BS07] §6.2, namely Lemmata, Corollaries and Propositions 6.2.5-6.2.10 merely rely on the results we just proved and do not involve any details about the exact definition of the graph X, thus their proofs need not be repeated here.

In the same vein we can adapt the proofs of the Lemmata in [BS07] §6.3 to the quasimetric setting. It then follows from [BS07] Thms. 6.3.1, 6.4.1, (cf. Thm. 74 below) that $\partial_\infty^{1/r}\mathrm{Hyp}_r(Z, \rho)$ is bilipschitz equivalent to (Z, ρ). So far this only holds for $r \leq 1/K^3$. Now the boundaries at infinity come equipped with a family of quasimetrics $a^{-(\cdot|\cdot)}$ for $a > 1$. The corresponding situation for hyperbolic approximations is that they should be taken for a family of parameters $r \in (0, 1)$, not just for $r \in (0, 1/K^3]$. Even though it should intuitively be possible to make a similar construction with balls as above, it seems the resulting graph is too difficult to control. For this reason, we resort to a scaling trick.

First of all we find it convenient to use $r = 1/K^3$ as a fixed reference for r.

Definition 72. *Let (Z, ρ) a complete K-quasi-metric space and $r \in (0, 1)$. Let $l(r, K) := \log_r(1/K^3) = -\frac{\log K^3}{\log r}$.*

The hyperbolic approximation of (Z, ρ) with parameter r, Hyp_r^K, is defined to be the graph of the hyperbolic approximation of $(Z, \rho^{1/l})$ with parameter r as described above, but scaled so that each edge has length $l = l(r, K)$.

Remark 73. *The graph $\mathrm{Hyp}_r(Z, \rho)$ does not depend on the choice of vertex system $\mathcal{V}$.*

Also, it follows from the bilipschitz Extension Theorems 75, 143 that the hyperbolic approximation is independent of the quasimetric constant K used, i.e. $\mathrm{Hyp}_r^K(Z, \rho)$ is roughly isometric to $\mathrm{Hyp}_r^{K'}(Z, \rho)$ if ρ is both a K- and a K'-quasimetric on Z. Furthermore, these same Extension Theorems immediately yield that approximations w.r.t. different parameters r, r' are merely scalings of each other, more precisely

$$\mathrm{Hyp}_r(Z, \rho) = \frac{\ln r}{\ln r'}\mathrm{Hyp}_{r'}(Z, \rho).$$

We refer to Appendix A for the proofs of these facts.

The following fundamental theorem summarizes the properties of the hyperbolic approximation.

Theorem 74 (Compare [BS07] Thms. 6.3.1, 6.4.1)**.** *Let (Z,ρ) be a complete quasimetric space, $r \in (0,1)$. The hyperbolic approximation $\mathrm{Hyp}_r(Z)$ is a visual geodesic hyperbolic space and there is a canonical identification $\partial_\infty \mathrm{Hyp}_r(Z) = Z$ of sets. Moreover, if (Z,ρ) is extended then for any $b \in \mathcal{B}(\omega)$, $\partial_\infty^{1/r,b}\mathrm{Hyp}_r(Z,\rho)$ and (Z,ρ) are bilipschitz equivalent. If (Z,ρ) is not extended, then $\partial_\infty^{1/r,o}\mathrm{Hyp}_r(Z,\rho)$ and (Z,ρ) are bilipschitz equivalent.*

The moral of the story is that, given a complete quasimetric space (Z,ρ), there is for every $a > 1$ exactly one (up to rough isometry) visual geodesic hyperbolic space X such that $\partial_\infty^a X$ is bilipschitz-quasimoebius to (Z,ρ), and the "functor" $\mathrm{Hyp}_{1/a}$ (in Ch. 7 we make this more precise) spits out exactly this space X when applied to (Z,ρ).

In the case of extended Z, $\mathrm{Hyp}(Z)$ has a distinguished boundary point ω corresponding to the infinitely remote point ξ of Z, while in the non-extended case the root o of the approximation will serve as distinguished base point.

Chapter 6

Extension Theorems

6.1 Extension Theorem for Bilipschitz Maps

The following theorem is well-known in the metric setting, see e.g. [BS07] Theorem 7.1. The proof in our quasimetric setting is exactly the same as in the metric setting of [BS07].

Theorem 75. *Let X be a visual and X' be a geodesic hyperbolic space, $o \in X, o' \in X'$. Then to every bilipschitz map $f : \partial_\infty^{a;o}, X \to \partial_\infty^{a;o'} X'$, there exists a roughly isometric map $F : X \to X'$ with $\partial_\infty F = f$.* □

Corollary 76 ([BS07] Corollaries 7.1.5, 7.1.6 and [BS00] Thm. 8.2)**.** *Let X be a visual hyperbolic space and $o \in X$, $a > 1$, $r \in (0, 1)$.*

X embeds roughly homothetically into $\mathrm{Hyp}_r \partial_\infty^{a;o} X$. If X is also roughly geodesic, then there is a rough homothety of X onto $\mathrm{Hyp}_r \partial_\infty^{a;o} X$.

In addition, X embeds roughly isometrically into $\mathrm{Hyp}_{1/a} \partial_\infty^{a;o} X$. If X is also roughly geodesic, then X is roughly isometric to $\mathrm{Hyp}_{1/a} \partial_\infty^{a;o} X$. □

Remark 77. *An analogous theorem can be stated for the case when the boundaries are equipped with extended quasimetrics with respect to some Busemann functions, see Thm. 143.*

When we are only concerned about the quasi-isometry class of the approximation, it is thus not necessary to specify the parameter r in $\mathrm{Hyp}_r(Z)$. Whenever we write only $\mathrm{Hyp}(Z)$ in a statement, it is to be understood that the statement is true for *every* $r \in (0, 1)$.

6.2 Extension of P-QS Maps

In this section we prove

Theorem 78 (Compare [BS00] Thm. 7.4)**.** *Let (Z, ρ), (Z', ρ') be two bounded complete quasi metric spaces and $\mathrm{Hyp}(Z)$, $\mathrm{Hyp}(Z')$ be their hyperbolic approximations. Suppose $f : Z \to Z'$ is a power quasisymmetric homeomorphism, i.e. η-QS with $\eta(t) = C \max\{t^\alpha, t^{1/\alpha}\}$ for some $C > 0, \alpha \geq 1$. Then there exists a power quasi-isometry $F : \mathrm{Hyp}(Z) \to \mathrm{Hyp}(Z')$ with $\partial_\infty F = f$.*

This theorem is trivial for $Z = \{z\}$, so we assume $|Z| \geq 2$. For convenience we shall also assume throughout this section that both spaces are K-quasimetric and that the approximations of both spaces are done w.r.t the *same* parameter $r = 1/(2K^3)$. This poses no loss of generality by Theorem 74, Corollary 76 and indepence of K of the hyperbolic approximation.

We assume the vertex system $\mathcal{V} = \{V_k\}$ is hereditary. We will split up the vertices into two disjoint subsets. Recall that if $v \in V_k$, then to v is associated the ball $B(v) = B_k(v) := B_{Kr^k}(\pi(v)) \subset Z$. Also, for any vertex v of a hyperbolic approximation, the notation $r(B(v))$ refers to the (abstract) radius of the ball associated to v, i.e. $r(B(v)) := Kr^{\ell(v)}$.

Definition 79. *A vertex $v \in V_k$ is called* regular *if the annulus $B_{Kr^k}(\pi(v)) \setminus B_{Kr^{k+1}}(\pi(v))$ is non-empty. It is called* singular *if it is not regular.*

The root o of a truncated hyperbolic approximation is always regular unless $Z = \{z\}$, which we assume is not the case.

Lemma 80. *If $v \in V_k$ is singular and connected radially to a vertex $w \in V_{k+1}$ and $\pi(w) \neq \pi(v)$, then w is regular and so is $(\pi(v), k+1) \in V_{k+1}$.*

Moreover, if w is a horizontal neighbour of $v \in V_k$, then at least one of v, w is regular.

Proof. $B(w) \subset B(v)$ by definition of radial edges. Since v is singular, this means $B(w) \subset B_{Kr^{k+1}}(\pi(v))$. On the other hand, $\rho(\pi(v), \pi(w)) \geq r^{k+1} > Kr^{k+2}$, which means w and $(\pi(v), k+1)$ are regular.

If $v, w \in V_k$ are both singular, then $\rho(v, w) \geq r^k$ and for any $z \in B(v)$, $\rho(v, z) < Kr^{k+1}$ by singularity of v. Hence $\rho(w, z) \geq r^k/K > Kr^{k+1}$ and z is not in $B(w)$, hence $B(v) \cap B(w) = \emptyset$. □

Remark 81. *If $v \in V_k$ and $B_{Kr^{k-1}}(v) \supsetneq B_{Kr^k}(v)$, $(\pi(v), k-1)$ may or may not be in V_{k-1}. At any rate, we know by maximality of V_{k-1} that there exists $w \in V_{k-1}$ which is radially connected to $v \in V_k$ and $\rho(\pi(w), \pi(v)) < r^{k-1}$.*

We will now define the map F of Theorem 78. The idea is to define it first on all regular vertices and then "fill in" the rest. First of all note the

Lemma 82. *For any vertex $v \in V_k$ of a hereditary vertex system $\mathcal{V}$ exactly one of the following holds.*

1. *v is regular,*
2. *v is singular and so are $v \in V_{k+l}$ for $0 \leq l < N$, while $v \in V_{k+N}$ is regular. $N \geq 1$,*
3. *v is singular in V_{k+l} for all $l \geq 0$.*

Proof. The notation $v \in V_{k+l}$ is meant to denote the element $(\pi(v), k+l)$ of V_{k+l}. The cases are mutually exclusive and exhaustive, so the lemma is evident. □

We will refer to the numbers in Lemma 82 as the *types* of a given vertex $v \in V_k$, type I vertices being the regular vertices and so on, cf. Fig. 6.1.

Definition 83. *If $v \in \mathcal{V}$ is regular, $F(v)$ is defined to be a vertex $v' \in \mathrm{Hyp}\,(Z')$ of highest level such that $B(v') \supset f(B(v))$.*

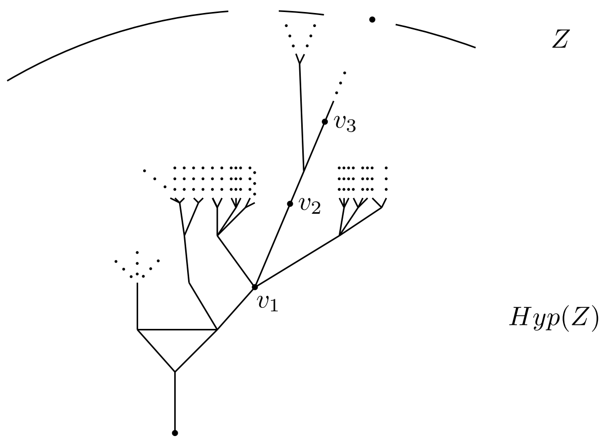

Figure 6.1: v_1, v_2, v_3 are vertices of type I, II, III respectively.

This defines F on the set of regular vertices up to an error of at most 1, as any two such vertices v' are evidently connected by an edge.

Lemma 84. *If $v \in V_k$ is regular, then so is $F(v)$.*

Proof. Denote by m the level $l(F(v))$ of $F(v)$ in $\mathrm{Hyp}\,(Z')$. If $F(v)$ were singular, $B_{Kr^m}(\pi(F(v))) \setminus B_{Kr^{m+1}}(\pi(F(v)))$ would have to be empty. This would mean that all of $f(B(v))$ would already be contained in $B_{Kr^{m+1}}(\pi(F(v)))$, contradicting the maximality of the level of $F(v)$ among all vertices containing $f(B(v))$. $\square$

Now suppose $v \in V_k$ is of type II. As noted before, v is not the root of $\mathrm{Hyp}\,(Z)$. In particular, there will be an $m \in \mathbb{N}$ and a $w \in V_{k-m}$ such that w is regular, $v \in V_{k-m+1}$ and singular, and $w \in V_{k-m}$ is radially connected to $v \in V_{k-m+1}$. $\pi(w)$ may or may not be equal to $\pi(v)$, confirm Remark 81. Trivially, all the v's on adjacent levels are radially connected. We define the following terms.

Definition 85. *A geodesic segment in* $\mathrm{Hyp}\,(Z)$ *through vertices $v_0, \ldots, v_N$ is called* singular *if the vertices v_1 up to and including v_{N-1} are all singular.*

By Lemma 80 and the paragraph following this definition, we may assume that all edges $v_0v_1, \ldots, v_{N-1}v_N$ are radial and that $\pi(v_1) = \ldots = \pi(v_N)$. It follows that the level function is monotonous along the geodesic and, after possibly reversing the order, we may assume

$$k - m = \ell(v_0) \leq \ell(v_1) < \ell(v_2) \ldots < \ell(v_{N-1}) \leq \ell(v_N) = k + l.$$

If v_0 is regular, we call it the lower end *of the singular geodesic and if v_N is regular, it is the* upper end *respectively.*

Every singular geodesic segment has a lower end since the root is regular. A singular geodesic with no upper end is called a singular ray*. The lower end of a singular ray is also called its* root*.*

Lower and, if they exist, upper ends are uniquely determined by the singular segment up to error 1. In particular, if $v_N \in V_N$ is an upper end we may assume

$\pi(v_N) = \pi(v_{N-1})$ because if v_{N-1} is singular, then $(\pi(v_{N-1}), N)$ is connected to v_N and both are regular. Similarly, if v_0 is a lower end and $\ell(v_0) = \ell(v_1)$, then any $w \in V_{\ell(v_o)-1}$ with $\rho(w, v_1) < r^{\ell(v_0)-1}$ is regular and we can replace v_0 with w. We may thus suppose $\ell(v_0) < \ell(v_1) < \ldots < \ell(v_N) = k + l$ With these assumptions, a vertex $v \in V_k$ of type II thus gives rise to a singular geodesic segment $wv_{k-m+1} \cdots v_k \cdots v_{k+l}$ with lower end $w \in V_{k-m}$ and upper end $v \in V_{k+l}$, where $\pi(v_{k-m+1}) = \ldots = \pi(v_{k+l})$ and every edge of which is radial, cf. Figure 6.2.

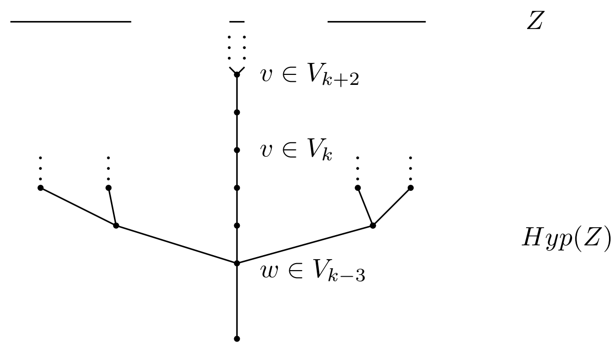

Figure 6.2: Singular geodesic $w \cdots v_{k+2}$ associated to $v_k = v \in V_k$.

The hope is now that $F(w)$ and $F(v_{k+l})$ will be joined in $\mathrm{Hyp}\,(Z')$ by a singular segment whose length is in bilipschitz correspondence to $|wv_{k+l}| = m+l$. This turns out to be roughly true, cf. Lemmata 88 and 89.

Lemma 86. *Suppose $v_k \in V_k$ is of type II and $v_{k+l} \in V_{k+l}$, $w \in V_{k-m}$ are the upper and the lower ends of the singular geodesic associated to $v_k \in V_k$. There is a $C_1 = C_1(\eta, K, r)$ such that if $l + m > C_1$, then $B(F(v_{k+l})) = f(B_{k+l}(v))$.*

More informally; the smallest ball containing $f(B_{k+l}(v))$ contains nothing besides $f(B_{k+l}(v))$.

Proof. Suppose $f(z) \in \mathrm{Im}\, f(Z) = Z'$ is outside of $f(B(v_{k+l}))$. So $z \notin B(v_{k+l})$ and therefore $\rho(v, z) \geq Kr^{k-m+1}$. Consequently we have for all $z' \in B(v_{k+l})$

$$\begin{aligned}
\frac{\rho(v_{k+l}, z')}{\rho(v_{k+l}, z)} &< r^{l+m-1}, \\
\frac{\rho'(f(v_{k+l}), f(z'))}{\rho'(f(v_{k+l}), f(z))} &< Cr^{\frac{1}{\alpha}(l+m-1)}, \\
\frac{\operatorname{diam} f(B(v_{k+l}))}{\rho'(f(v_{k+l}), f(z))} &< KCr^{\frac{1}{\alpha}(l+m-1)}, \\
\frac{r(B(F(v_{k+l})))}{\rho'(f(v_{k+l}), f(z))} &< \widetilde{C}r^{\frac{1}{\alpha}(l+m)} \qquad \text{by regularity of } v_{k+l}.
\end{aligned}$$

Since $\widetilde{C}$ is a uniform constant depending on η, K and r only, there is a C_1 such that if $l + m > C_1$, we will have

$$r(B(F(v_{k+l}))) < \frac{1}{K}\rho'(f(v_{k+l}), f(z)). \tag{6.1}$$

But of course

$$\begin{aligned}\rho'(f(v_{k+l}), f(z)) &\le K \max\{\rho'(f(v_{k+l}), F(v_{k+l})),\ \rho'(f(z), F(v_{k+l}))\} \\ &\le K \max\{r(B(F(v_{k+l}))),\ \rho'(f(z), F(v_{k+l}))\}.\end{aligned}$$

This and (6.1) imply

$$\rho'(f(z), F(v_{k+l})) > r(B(F(v_{k+l}))).$$

□

Corollary 87. *The center $\pi(F(v_{k+l}))$ of $F(v_{k+l})$ is in $f(B(v_{k+l}))$.*

Now we want to verify that the image of the upper end of a singular geodesic is the upper end of a singular geodesic with comparable length.

Lemma 88 (Upper Ends go to Upper Ends)**.** *Suppose $v_k \in V_k$ is of type II and $v_{k+l} \in V_{k+l}$, $w \in V_{k-m}$ are the upper and the lower ends respectively of the singular geodesic associated to v_k. There exists a uniform constant $C_2 = C_2(C, C_1, K, \eta, r)$ such that $F(v_{k+l})$ is the upper end of a singular geodesic in* Hyp (Z') *whose length L' satisfies*

$$\frac{1}{\alpha}(m+l) - C_2 \le L' \le \alpha(m+l) + C_2.$$

Proof. Let $z \in Z \setminus B_{k+l}(v)$. Then $\rho(z, \pi(v)) \ge K r^{k-m+1}$. First of all take $C_2 \ge C_1$. Then by Corollary 87, $\exists \hat{v} \in B_{k+l}(v)$ such that $f(\hat{v}) = \pi(F(v_{k+l}))$. Now for all $z_1 \in \overline{B_{k+l}}(v)$, $z_2 \in Z \setminus B_{k+l}(v)$ we have

$$\rho(\hat{v}, z_1) < K^2 r^{k+l}, \qquad \rho(\hat{v}, z_2) \ge r^{k-m+1}.$$

Thus

$$\frac{\rho(\hat{v}, z_1)}{\rho(\hat{v}, z_2)} < K^2 r^{l+m-1}, \text{ whence}$$
$$\frac{\rho'(f(\hat{v}), f(z_1))}{\rho'(f(\hat{v}), f(z_2))} < C K^{2/\alpha} r^{\frac{1}{\alpha}(l+m-1)},$$

which, since $r(B(F(v_{k+l}))) \asymp_r \operatorname{diam}(f(B_{k+l}(v))) \asymp_K \sup_{z_1} \rho'(f(\hat{v}), f(z_1))$, gives

$$\frac{r(B(F(v_{k+l})))}{\rho'(f(\hat{v}), f(z_2))} < D r^{\frac{1}{\alpha}(l+m-1)} = \widetilde{C_2} r^{\frac{1}{\alpha}(l+m)}.$$

From this it follows that $(f(\hat{v}), p-q) \in V'_{p-q}$ for all $0 \le q \le \frac{1}{\alpha}(l+m) - \widetilde{C_2}$, and it is obviously singular on all these levels.

On the other hand, v_{k+l} is regular, meaning there exists a $z_3 \in B_{k+l}(v)$ with $\rho(\hat{v}, z_3) \ge r^{k+l+1}$. With $z_2 \in Z \setminus B_{k+l}(v)$ such that $\rho(\hat{v}, z_2) \le K^2 r^{k-m}$ (exists since $w \in V_{k-m}$ is regular and $\hat{v} \in B(v_{k+l}) \subset B(w)$), we have

$$\frac{\rho(\hat{v}, z_2)}{\rho(\hat{v}, z_3)} \le (K^2/r) \cdot r^{-(m+l)},$$

that is,

$$\frac{\rho'(f(\hat{v}), f(z_2))}{\rho'(f(\hat{v}), f(z_3))} \le C(K^2/r)^\alpha \cdot r^{-\alpha(m+l)},$$

which bounds the length of the singular geodesic descending from $F(v_{k+l})$ by $\alpha(m+l) + \widehat{C_2}$. Setting $C_2 := \max\{C_1, \widetilde{C_2}, \widehat{C_2}\}$ proves the lemma. □

So if $wv_{k-m+1}\cdots v_k\cdots v_{k+l}$ is a singular geodesic in $\mathrm{Hyp}\,(Z)$, then $F(v_{k+l})$ is the upper end of a singular geodesic in $\mathrm{Hyp}\,(Z')$ with controlled length. Now we want to know how $F(w)$ and the lower end of the image singular geodesic are related, cf. Fig. 6.3.

Lemma 89 (Lower Ends go roughly to Lower Ends)**.** *Suppose $v_k \in V_k$ is singular. If v_k is of type II, let $wv_{k-m+1}\cdots v_k\cdots v_{k+l}$ be the singular segment in* $\mathrm{Hyp}\,(Z)$ *determined by v_k with lower end $w \in V_{k-m}$ and upper end $v_{k+l} \in V_{k+l}$, and let $w'v'_{p-q+1}\cdots F(v_{k+l})$ be the singular segment in* $\mathrm{Hyp}\,(Z')$ *associated to $F(v_{k+l}) \in V'_p$ according to Lemma 88. If v_k is of type III and $wv_{k-n}\cdots v_k\cdots$ the associated singular ray in* $\mathrm{Hyp}\,(Z)$*, denote by w' the root of the singular ray in* $\mathrm{Hyp}\,(Z')$ *associated to $f(\pi(v_k))$.*

There is a uniform constant $C_3 = C_3(\eta, K, r)$ such that $|w'F(w)| \leq C_3$.

Proof. We show it first for v of type II. We may assume that $l+m > C_1$, for if not, Lemma 88 says that w' is uniformly close to $F(v_{k+l})$, and the fact that $\mathrm{diam} f(B(v_{k+l}))$ is uniformly comparable to $\mathrm{diam} f(B(w))$ (and the sets intersect) shows that $F(w)$ uniformly close to $F(v_{k+l})$.

Now $F(w)$ is by definition the smallest ball containing $f(B_{k-m}(w))$. In particular $f(B_{k+l}(v)) \subset B(F(w))$, so that $B(F(w)) \cap B(w') \neq \emptyset$. Now the distance between vertices whose associated balls intersect is roughly equal to their level distance (cf. [BS07], Lemma 6.2.7). Hence we must show that $l(w') \doteq l(F(w))$, which is the case iff $r(B(w')) \asymp r(F(w))$, iff

$$\mathrm{diam}\, f(B_{k-m}(w)) \asymp r(B(w')). \tag{6.2}$$

Now,

$$r(B(w')) \asymp_r \inf_{z' \in Z' \setminus f(B_{k+l}(v))} \rho'(z', \pi(F(v_{k+l}))).$$

But we know (Cor. 87) that the center of the ball $F(v_{k+l})$ is given by $f(\hat{v})$ for some $\hat{v} \in B_{k+l}(v)$. Since f is bijective we can write

$$r(B(w')) \asymp_r \inf_{z \in Z \setminus B_{k+l}(v)} \rho'(f(z), f(\hat{v})). \tag{6.3}$$

For the l.h.s. of (6.2) we have

$$\mathrm{diam}\, f(B_{k-m}(w)) \asymp_K \sup_{z \in B_{k-m}(w)} \rho'(f(\hat{v}), f(z)), \tag{6.4}$$

because $\hat{v} \in B_{k+l}(v) \subset B(w)$.

With (6.3) and (6.4), (6.2) becomes

$$\sup_{z \in B_{k-m}(w)} \rho'(f(\hat{v}), f(z)) \asymp \inf_{z \in Z \setminus B_{k+l}(v)} \rho'(f(\hat{v}), f(z)) \tag{6.5}$$

Simplifying further, for any $z \in B(w) \setminus B(v_{k+l})$ we have $\frac{\rho(v,\hat{v})}{\rho(v,z)} < 1$, whence by quasi-symmetry

$$\sup_{z \in B(w)} \rho'(f(\hat{v}), f(z)) \leq D \cdot \sup_{z \in B(w) \setminus B(v_{k+l})} \rho'(f(\hat{v}), f(z)),$$

for a uniform D. This gives

$$\sup_{z \in B(w)} \rho'(f(\hat{v}), f(z)) \asymp \sup_{z \in B(w) \setminus B(v_{k+l})} \rho'(f(\hat{v}), f(z)). \tag{6.6}$$

Likewise we get

$$\inf_{z\in Z\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)) \asymp \inf_{z\in B(w)\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)), \tag{6.7}$$

for pick a $\hat{z}\in B(w)\setminus B(v_{k+l})$ such that for some uniform E

$$\frac{\rho(\hat{v},\hat{z})}{\rho(\hat{v},z)} \le E \quad \forall z\in Z\setminus B(v_{k+l}).$$

Then

$$\eta(E)\cdot \inf_{z\in Z\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)) \ge \rho'(f(\hat{v}),f(\hat{z})) \ge \inf_{z\in B(w)\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)),$$

from which (6.7) follows immediately.

With (6.6) and (6.7), (6.5) follows if we prove

$$\inf_{z\in B(w)\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)) \asymp \sup_{z\in B(w)\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)).$$

One direction is trivial and we just have to show

$$\sup_{z\in B(w)\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)) \le H\cdot \inf_{z\in B(w)\setminus B(v_{k+l})} \rho'(f(\hat{v}),f(z)), \tag{6.8}$$

for some uniform constant H. But in fact, for any $z\in B(w)\setminus B(v_{k+l})$ we have $r^{k-m+1}\le \rho(\hat{v},z)\le K^2 r^{k-m}$, thus there is a uniform $\widetilde{H}$ such that

$$\frac{\rho(\hat{v},z_1)}{\rho(\hat{v},z_2)} \le \widetilde{H}\ \forall z_1,z_2\in B(w)\setminus B(v_{k+l}), \text{ and hence}$$

$$\frac{\rho'(f(\hat{v}),f(z_1))}{\rho'(f(\hat{v}),f(z_2))} \le \eta(\widetilde{H}).$$

This implies (6.8) and thereby the lemma for v_k of type II.

The argument for v_k of type III is analoguous. $B(w')$ and $B(F(w))$ again intersect, so we must estimate their level difference. Denote $\hat{z}:=\pi(v_k)$. $F(w)$ is the smallest ball containing $f(B(w))$, while the radius of $B(w')$ is determined by when a ball around $f(\hat{z})$ starts to contain points in $Z'\setminus\{f(\hat{z})\}$.

In formulas

$$r(B(F(w))) \asymp_{C(K,r)} \operatorname{diam} f(\overline{B}(w)) \qquad r(w') \asymp_{D(K,r)} \inf_{z\in Z\setminus\{\hat{z}\}} \rho'(f(z),f(\hat{z})),$$

where $C(K,r)$ and $D(K,r)$ are appropriate expressions involving only K and r. Since $\operatorname{diam} f(B(w)) \asymp_{E(r,K)} \sup_{z\in B(w)} \rho'(f(\hat{z}),f(z))$, the claim follows once we show

$$\sup_{z\in B(w)} \rho'(f(\hat{z}),f(z)) \asymp_{\widetilde{C_4}(K,r)} \inf_{z\in Z\setminus\{\hat{z}\}} \rho'(f(\hat{z}),f(z)). \tag{6.9}$$

Now the same steps as in the proof of (6.5) yield the lemma for v_k of type III. ☐

So far we have only defined where F maps regular vertices. We are now in a position to extend the domain of F to all of $\operatorname{Hyp}(Z)$.

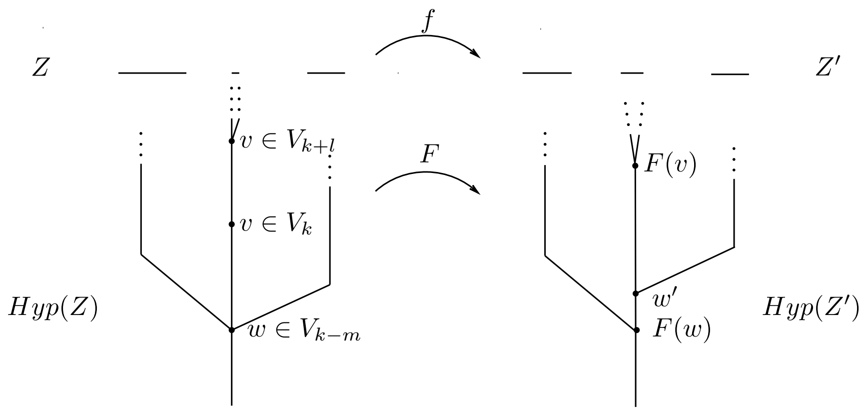

Figure 6.3: The distance between w' and $F(w)$ is uniformly bounded.

$v \in V_k$ **is of type** I $F(v) \in \mathrm{Hyp}\,(Z')$ is defined to be a vertex of highest level w' such that $f(B(v)) \subset B(w')'$.

$v \in V_k$ **is of type** II $v = v_k \in V_k$ lies on a singular geodesic $wv_{k-m+1} \cdots v_{k+l}$ with lower and upper ends $w \in V_{k-m}$, $v = v_{k+l} \in V_{k+l}$. In case $l + m < \alpha C_2$, set $F(v) := F(w)$. If $l + m \geq \alpha C_2$, then $F(v_{k+l}) \in V'_p$ is the upper end of a singular geodesic whose length L' satisfies $\frac{1}{\alpha}(l+m) - C_2 \leq L' \leq \alpha(l+m) + C_2$ (Lemma 88) and if $w' \in V'_{p-L}$ denotes the lower end of this singular geodesic, then $|F(w)w'| \leq C_3$ (Lemma 89). Let $L = l + m$. In this case define $F(v \in V_k)$ to be a vertex v' on the singular geodesic from w' to $F(v_{k+l})$ for which $|w'v'| \doteq_1 \frac{L'}{L}|wv|$.

$v \in V_k$ **is of type** III $v \in V_k$ lies on a singular ray in $\mathrm{Hyp}\,(Z)$ going to $\pi(v) \in Z$. Since $|Z| \geq 2$, this singular ray has a regular lower end $w \in V_{k-m}$. Since f is a homeomorphism, $f(v)$ is isolated in Z', thus there is a singular ray in $\mathrm{Hyp}\,(Z')$ starting at some regular $w' \in V'_p$. $F(v)$ is defined as the (unique) vertex $v' \in V'_{p+m}$ on this ray. Equivalently, $F(v)$ is the vertex v' on the singular ray in $\mathrm{Hyp}\,(Z')$ which has the same distance from w' as v has from w.

This defines F on the whole vertex set $\mathcal{V}$, and up to a rough isometry, F is then well-defined on all of $\mathrm{Hyp}\,(Z)$.

Theorem 90. *The map* $F : \mathrm{Hyp}\,(Z) \to \mathrm{Hyp}\,(Z')$ *described above is a quasi-isometry, and* $\partial_\infty F = f$.

Proof. We first show that F is Lipschitz. Since $\mathrm{Hyp}\,(Z)$ is geodesic, this follows if we show that the distance $|F(v)F(w)|$ is uniformly bounded for neighboring $v, w \in \mathrm{Hyp}\,(Z)$. Now if v, w are both of type I, it follows by standard arguments (such as those used in the proof of Theorem 7.2.1 in [BS07]) that the level difference of $F(v)$ and $F(w)$ is uniformly bounded. If, w.l.o.g. v is of type I and w of type II, Lemmas 88 and 89 (or the definition of F if w is not on a long enough singular geodesic) imply that $|v'w'|$ uniformly bounded. If v of type I and w of type III, Lemma 89 does the job. A vertex of type II never neighbors a vertex of type III. This proves that F is Lipschitz.

Next define a map $G : \mathrm{Hyp}\,(Z') \to \mathrm{Hyp}\,(Z)$ corresponding to $f^{-1} : Z' \to Z$ in the same way F was defined (and with the same choice of vertex systems $\mathcal{V}$, $\mathcal{V}'$). Of course G is then also Lipschitz. We show $G \circ F \doteq \mathrm{id}_{\mathrm{Hyp}\,(Z)}$.

v of type I: By definition $B(G \circ F(v)) \supset B(v)$. In particular, the balls intersect. Their distance is uniformly bounded iff the diameters of these sets are uniformly comparable. But this follows from the facts that $f(B(v)) \subset B(F(v))$, $\mathrm{diam}\, f(B(v))$ is uniformly comparable to $\mathrm{diam}\, B(F(v))$, and that f^{-1} is quasisymmetric. The doubtful reader is referred to [TV80] Thm. 2.5, which describes exactly this situation.

v of type II: We have a singular geodesic $wv_{k-m+1} \cdots v = v_k \cdots v_{k+l}$ with lower end $w \in V_{k-m}$ and upper end v_{k+l}. By Lemma 88 applied twice to F and then G, there is a uniform constant C_5 such that if $l+m > C_5$, not only $F(v_{k+l})$ is an upper end of a singular geodesic in $\mathrm{Hyp}\,(Z)$ but even $G(F(v_{k+l}))$ is still the upper end of singular geodesic in $\mathrm{Hyp}\,(Z)$. $F(v_{k+l})$, as usual, is a smallest ball containing $f(B_{k+l}(v))$. But by Lemma 86 $B(F(v_{k+l})) = f(B_{k+l}(v))$. In particular, $G(F(v_{k+l}))$, being the smallest ball containing $f^{-1}(B(F(v_{k+l})))$, is just $B_{k+l}(v)$. In other words, $G(F(v_{k+l})) = v_{k+l}$. By definition of F and G it is now obvious that $G(F(v))$ is uniformly close to v.

If the singular geodesic $wv_{k-m+1} \cdots v = v_k \cdots v_{k+l}$ is shorter than C_5, then v is in particular uniformly close to a type I vertex, namely w (or v_{k+l}). The Lipschitz property of F and G and the fact that $G(F(w))$ is uniformly close to w imply that $G(F(v))$ is uniformly close to v.

v of type III: $\pi(v) = z$, an isolated point in Z. $f(z)$ is an isolated point in Z' and by definition of F, the ray in $\mathrm{Hyp}\,(Z)$ associated to z, on which v lies, is mapped one-to-one onto the ray in $\mathrm{Hyp}\,(Z')$ associated to $f(z)$. But then G maps this ray back in one-to-one fashion to the ray associated to $f^{-1}(f(z)) = z$. So in this case we have in fact $v = G(F(v))$.

This proves $G \circ F \doteq \mathrm{id}_{\mathrm{Hyp}\,(Z)}$. Since the domain of G is all of $\mathrm{Hyp}\,(Z')$, it follows that $F(\mathrm{Hyp}\,(Z))$ is cobounded in $\mathrm{Hyp}\,(Z')$, thus F is a quasi-isometry.

It remains to show that $\partial_\infty F = f$. By [BS07], Thm. 5.2.17, we know that F *does* induce a homeomorphism $\partial_\infty F : Z \to Z'$. So take a sequence $\{v_i\}$ of vertices converging to $z \in Z$. We have $\pi(v_i) \to z$ in (Z, ρ). Since the limit of the sequence $\{F(v_i)\}$ does not depend on the representative $\{v_i\} \in z$, we may take the latter such that $B(v_{i+1}) \subset B(v_i)$ (cf. [BS07], Lemma 6.3.2) Then $\{F(v_i)\}$ converges to some $z' \in Z'$. In particular, $l'(F(v_i)) \overset{i\to\infty}{\longrightarrow} \infty$. Since $\rho'(f(\pi(v_i)), \pi(F(v_i))) \le Kr^{l'(F(v_i))}$ and $f(\pi(v_i)) \to f(z)$, we get $\pi(F(v_i)) \to f(z)$ in Z' and this implies $\partial_\infty F(z) = f(z)$. $\square$

The following example shows that an arbitrary quasisymmetric map (i.e. not necessarily P-QS) between boundaries of visual geodesic hyperbolic spaces need not induce a quasi-isometric map between the spaces.

Example 91. *Take as X the rooted tree with root o that has a ray going straight to infinity from which rays branch off at levels $\log(n!)$ for each n. Take as Y the same space (with root o') but the side rays branch off at levels $n!$. Then the boundaries are quasi-symmetrically equivalent because the map*

$$f : \{\exp(-n!)\,|n \in \mathbb{N}\} \to \{1/n!\,|\,n \in \mathbb{N}\} \tag{6.10}$$

$$\exp(-n!) \mapsto 1/n! \tag{6.11}$$

is quasi-symmetric (cf. [Hei01] p. 89). However, the spaces X and Y, though visual and geodesic, are not quasi-isometric. Too see this, look at tripods with center u a knot in X, x and y on the segments to the previous and succeeding knot respectively and z on the ray that branches off from the knot u.

Then $cd(x, y, z, u) = 0$ (in fact, so is any permutation). Therefore, since quasi-isometric maps between geodesic spaces are strictly P-QI, $< x', y', z', u' > \doteq 0$ up to some constant. This means that the image must look like a cross $\times$, but since the map also preserves Gromov products, the image is in fact again a tripod. This implies the following

Fact : *Knots of X (which are at levels $\log(n!)$ are mapped to within bounded distance of knots in Y, which are at levels $n!$.*

But this would imply that we have a quasi-isometric map from $\{\log(n!) \,|\, n \geq 2\}$ to $\{n! \,|\, n \geq 2\}$, which is clearly not possible.

6.3 Extension of Inversions

There are good reasons why one would not be satisfied with describing the quasisymmetric structure of the boundary, but would rather have a result on its quasi*moebius* structure. Namely, there is in general no uniform constant L such that $\mathrm{id} : \partial_\infty^{a,o} X \to \partial_\infty^{a,o'} X$ is L-bilipschitz for any $o, o' \in X$. However, there *is* a uniform L (depending on a, δ) such that it is L-bilipschitz-quasi*moebius*. In other words, the ratio of a triple of boundary points is not a uniform quantity, whereas the cross-ratio of a quadruple is. For more on this we refer to [Sch09] Thm. 8.1. Note also that that Paulin [Pau96] showed that in hyperbolic groups the induced boundary maps of the left translations $L_g : G \to G$ are uniformly quasimoebius. They are, however, not uniformly quasisymmetric. All this motivates us to look for an extension theorem for quasimoebius maps in the spirit of the Poincaré extension theorems for classical hyperbolic space.

In this section we prove that the hyperbolic approximation of a bounded quasimetric space (Z, ρ) is roughly isometric to the hyperbolic approximation (with the same parameters) of the extended quasimetric space (Z, ρ') where ρ' is the inversion at a point in Z of ρ. This result will be combined with Theorem 78 to give the desired quasimoebius extension.

Theorem 92. *Let (Z, ρ) be a bounded complete quasi-metric space and ρ' the quasi-metric obtained from ρ by inversion in a point $\omega \in Z$,*

$$\rho'(a, b) := \frac{\rho(a, b)}{\rho(a, \omega)\rho(b, \omega)}.$$

Then the (truncated) hyperbolic approximation of (Z, ρ) is roughly isometric to the hyperbolic approximation of (Z, ρ'). More precisely, for every $r \in (0, 1)$ there exists a rough isometry $F : \mathrm{Hyp}_r(Z, \rho) \to \mathrm{Hyp}_r(Z, \rho')$ that induces the identity in $\partial_\infty \mathrm{Hyp}\,(Z) = Z$.

This theorem is trivial for $Z = \{z, \omega\}$, so we shall assume $|Z| \geq 3$.

Remark 93. *The proof of this Theorem basically consists of a series of uniform comparability statements, $\cdot \asymp \cdot$, all of which remain true if the boundary quasimetrics are replaced by ones that are bilipschitz equivalent to them. In* *particular, the theorem allows us to conclude, via the bilipschitz extension Theorem 75, that* $\mathrm{Hyp}_{1/a}(\partial_\infty^{a,b_1(\omega)}X)$ *is roughly isometric to* $\mathrm{Hyp}_{1/a}(\partial_\infty^{a,b_2(\omega)}X)$*, where* $b_1(\omega), b_2(\omega)$ *are two arbitrary Busemann functions at* ω*. This fact will be needed in the proof of* $III) \Rightarrow I)$ *in Theorem 105.*

Note that if (Z,ρ) is K-quasimetric, then (Z,ρ') is K^2-quasimetric. Throughout this section we assume that both approximations $\mathrm{Hyp}\,(Z,\rho), \mathrm{Hyp}\,(Z,\rho')$ are done with respect to the same K. Since the rough isometry class of the approximations does not depend on the K used, this poses no danger. Moreover, we may assume $r = 1/K^3$, since for all other values of r, Hyp_r can be obtained by scaling the graphs $\mathrm{Hyp}_{r'}(Z,\rho)$, $\mathrm{Hyp}_{r'}(Z,\rho')$, where $r' = 1/K^3$, by the same factor.

In addition, it turns out to be advantageous to work with a special choice of vertex system $\mathcal{V}$ for $\mathrm{Hyp}\,(Z,\rho)$. Namely we require that $\mathcal{V}$ be hereditary and the root o be centered at the inversion point ω, $\pi(o) = \omega$. In particular, we then have a canonical "ray to ω" in $\mathrm{Hyp}\,(Z,\rho)$, namely the radial geodesic ray consisting of all vertices centered at ω. We will often refer to this ray as *the ray* $o\omega$.

The idea of the definition for F is to do the same as for quasi-symmetric maps whenever ω is not involved, and "invert the orientation" on the ray $o\omega$. This corresponds to the fact that the inversion restricted to $Z \setminus O$, where O is any neighborhood of ω, is a PQ-symmetry onto its image because it is a Moebius map between bounded spaces (cf. Lemma 103).

We define the map F.

Definition 94.

1. *If v is regular with $\pi(v) = \omega$ and $v \neq o$, set $F(v) :=$ any vertex w of highest level in* $\mathrm{Hyp}\,(Z,\rho')$ *such that $B^{\rho'}_{Kr^{l(w)}}(w)$ contains $B^{\rho}_{Kr^{l(v)+1}}(\pi(v))^c$.*
2. *If v is a horizontal neighbor to a vertex $\tilde{v}$ as in 1, set $F(v) := F(\tilde{v})$.*
3. *If $v \neq o$ is regular and neither as in 1 nor 2, set $F(v) :=$ any vertex w of highest level in* $\mathrm{Hyp}\,(Z,\rho')$ *such that $B^{\rho'}(w) \supset B^{\rho}_{Kr^{l(v)}}(\pi(v))$.*
4. *For the root o, if the immediate radial successor v to o on the ray $o\omega$ is regular, set $F(o) := F(v)$. If this v is not regular, then $Z \setminus B(v)$ is separated from the rest of Z (in the sense that the two sets have positive distance) and the same is the case in (Z,ρ'). Furthermore, there is a branch point (cf. [BS07], p. 72) in* $\mathrm{Hyp}\,(Z,\rho')$ *for $\{B := B^{\rho}_{Kr^{l(o)+1}}(\omega), Z \setminus B\}$. In this case set $F(o) =$ such a branch point.*
5. *If v is singular and lies on a singular segment w_1w_2 in* $\mathrm{Hyp}\,(Z,\rho)$*, map it to an appropriate vertex on the singular segment associated to w_1w_2 in* $\mathrm{Hyp}\,(Z,\rho')$*, cf. Lemma 99.*
6. *If v is singular and lies on a singular ray wz in* $\mathrm{Hyp}\,(Z,\rho)$*, map v to an appropriate vertex on the singular ray in* $\mathrm{Hyp}\,(Z,\rho')$ *associated to the ray wz, cf. Lemma 100.*

The verification that F is a rough isometry is straightforward but a bit tedious. We first show $|F(v)F(w)| \doteq |vw|$ for v, w from a cobounded subset of the set of regular vertices, Lemma 97. Then we can extend it to all v, w regular.

Afterwards we show well-behavedness of singular segments and rays, Lemmata 99 and 100 respectively.

Notation 95. *For v, w vertices in a hyperbolic approximation of a quasimetric space Z, we define the shorthand*

$$\sup \rho(z_v, z_w) := \sup_{\substack{z_v \in B(v) \\ z_w \in B(w)}} \rho(z_v, z_w)$$

Lemma 96. *Let v, w be any regular vertices in* $\mathrm{Hyp}\,(Z, \rho)$*, where* $\mathrm{Hyp}\,(Z, \rho)$ *has edge lengths equal to one. Then*

$$|vw| \doteq \log_r \Big(\frac{\mathrm{diam}(B(v))\mathrm{diam}(B(w))}{\sup \rho(z_v, z_w)^2} \Big).$$

Proof. There is a geodesic connecting v to w that has either exactly one or exactly two points of lowest level (cf. [BS07] Lemma 6.2.6). In either case, there is a branch point u for $\{v, w\}$ with distance at most one from any lowest level vertex. Then

$$|vw| \doteq_1 (l(v) - l(u)) + (l(w) - l(u)).$$

But $l(v) \doteq \log_r(\mathrm{diam} B(v))$ by regularity of v (the error constant depending on K, r), and the same for w.

Now $B(v) \cup B(w) \subset B(u)$ by definition. On the other hand, any vertex t such that $B(v) \cup B(w) \subset B(t)$ is uniformly close (error 1) to a cone point by [BS07] Lemma 6.2.1. Take t to be any vertex of highest level satisfying $B(v) \cup B(w) \subset B(t)$, then t is uniformly close to a (and hence, any) branch point. But then $\mathrm{diam}(B(t)) \asymp \sup \rho(z_v, z_w)$. The lemma follows. □

Lemma 97. *Let v, w be regular vertices in* $\mathrm{Hyp}\,(Z, \rho)$ *which if centered at ω or horizontally connected to $o\omega$ are at least two levels above the root. Then $|F(v)F(w)| \doteq |vw|$.*

Proof. For the proof we show that

$$\frac{\mathrm{diam}_{\rho'}(B^{\rho'}(F(v)))\mathrm{diam}_{\rho'}(B^{\rho'}(F(w)))}{\sup \rho'(z_{F(v)}, z_{F(w)})^2} \asymp \frac{\mathrm{diam}_{\rho}(B^{\rho}(v))\mathrm{diam}_{\rho}(B^{\rho}(w))}{\sup \rho(z_v, z_w)^2}, \quad (6.12)$$

which implies the claim by Lemma 96.

If $\pi(v) = \pi(w) = \omega$, we have $\mathrm{diam}_{\rho'}(B(F(v))) \asymp \frac{1}{\mathrm{diam}_{\rho}(B(v))}$. (6.12) simplifies to

$$\sup \rho'(z_{F(v)}, z_{F(w)}) \asymp \frac{\sup \rho(z_v, z_w)}{\mathrm{diam}(B(v))\mathrm{diam}(B(w))},$$

both sides of which compare uniformly to $1/\mathrm{diam}(B(v))$ (if w.l.o.g. $l(v) \geq l(w)$). By II of Definition 94, the same argument gives (6.12) for vertices horizontally connected to the ray $o\omega$. So suppose $\pi(v) = \omega$ and w not horizontally connected to the ray. Then

$$\rho'(z, w) = \frac{\rho(z, w)}{\rho(z, \omega)\rho(w, \omega)} \asymp \frac{\rho(z, w)}{\rho(w, \omega)^2} \quad \forall z \in B(w),$$

whence

$$\operatorname{diam}_{\rho'}(B(w)) \asymp \frac{\operatorname{diam}_\rho(B(w))}{\rho(w,\omega)^2}.$$

(6.12) then becomes

$$\frac{\operatorname{diam}_{\rho'}(B(F(v)))}{\sup \rho'(z_{v_{+1}^c}, z_w)^2 \rho(w,\omega)^2} \asymp \frac{\operatorname{diam}_\rho(B(v))}{\sup \rho(z_v, z_w)^2}, \tag{6.13}$$

where $z_{v_{+1}^c}$ suggests elements in $B(v_{+1})^c := B^\rho_{Kr^{l(v)+1}}(v)^c$. Since $\operatorname{diam}_{\rho'}(B(F(v))) \asymp 1/\operatorname{diam}_\rho(B(v))$, (6.13) is equivalent to

$$\sup \rho'(z_{v_{+1}^c}, z_w)\rho(w,\omega) \asymp \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v,\omega)}.$$

Thus we must show

$$\sup \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c},\omega)\rho(z_w,\omega)} \cdot \rho(w,\omega) \asymp \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v,\omega)}$$

which, since $\rho(z_w,\omega) \asymp \rho(w,\omega)$, finally becomes

$$\sup \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c},\omega)} \asymp \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v,\omega)}. \tag{6.14}$$

We prove (6.14), which implies the lemma in case v on the ray, w not horizontally connected to the ray. We show first that the l.h.s. of (6.14) is $\geq \frac{1}{K}$. Since v is regular $\exists z_1 \in B(v_{+1})^c$ with $\rho(\omega, z_1) < Kr^k$, and since v is at least 2 from the root, there also exists $z_2 \in B(v_{+1})^c$ with $\rho(\omega, z_2) \geq r^{k-1}$. Now suppose for all z_1 with $\rho(\omega, z_1) < Kr^k$, where $k = \ell(v)$.

$$\rho(z_1, z_w) < \frac{1}{K}\rho(z_1,\omega) \ \forall z_w \in B(w).$$

Then $\rho(z_1,\omega) \asymp_K \rho(z_w,\omega)$. But now z_2 is much farther from ω than z_1, hence

$$\rho(z_2,\omega) \asymp_K \rho(z_2, z_w) \ \ \forall z_w \in B(w).$$

This shows that the l.h.s. of (6.14) is $\geq 1/K$ in any case.

Next suppose

$$\sup \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c},\omega)} > K^4. \tag{6.15}$$

Then since necessarily $\rho(z_{v_{+1}^c}, z_w) \asymp_K \rho(z_w,\omega)$ for $z_{v_{+1}^c}, z_w$ such that the sup is (almost) attained,

$$\rho(z_w,\omega) > K^3 \rho(z_{v_{+1}^c},\omega). \tag{6.16}$$

That is, when $z_w, z_{v_{+1}^c}$ are taken so that the sup is (almost) attained, z_w will be much farther away from ω than $z_{v_{+1}^c}$. We want to know that then $z_{v_{+1}^c}$ may as well be taken in $B(v)$, thus we must show that if $z_2 \in B(v)$ is arbitrary, then the quantity

$$\frac{\rho(z_2, z_w)}{\rho(z_2,\omega)},$$

where the z_w is the same as above, is not smaller (or at least not by much) than when z_2 is replaced by $z_{v^c_{+1}}$. So pick $z_2 \in B(v)$ arbitrary. We may suppose $\rho(z_2, \omega) < \rho(z_{v^c_{+1}}, \omega)$, otherwise $z_{v^c_{+1}}$ would already be in $B(v)$ and we are done. So then

$$\rho(z_{v^c_{+1}}, z_2) \leq K\rho(z_{v^c_{+1}}, \omega) \overset{(6.16)}{<} \frac{1}{K^2}\rho(z_w, \omega) \leq \frac{1}{K}\rho(z_{v^c_{+1}}, z_w),$$

whence

$$\rho(z_{v^c_{+1}}, z_w) \asymp_K \rho(z_2, z_w).$$

Since $\rho(z_{v^c_{+1}}, \omega) > \rho(z_2, \omega)$ it thus follows that

$$\frac{\rho(z_2, z_w)}{\rho(z_2, \omega)} > \frac{1}{K}\frac{\rho(z_{v^c_{+1}}, z_w)}{\rho(z_{v^c_{+1}}, \omega)}.$$

It follows that the claimed uniform comparability of (6.14) holds.

It remains to prove (6.14) when

$$\frac{1}{K} \leq \sup \frac{\rho(z_{v^c_{+1}}, z_w)}{\rho(z_{v^c_{+1}}, \omega)} \leq K^4.$$

In fact we show more, namely

$$\sup \frac{\rho(z_{v^c_{+1}}, z_w)}{\rho(z_{v^c_{+1}}, \omega)} \asymp_{K^4} 1 \implies \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v, \omega)} \asymp 1. \tag{6.17}$$

The assumption on the l.h.s. means in particular that for any choice of $z_{v^c_{+1}}$, every z_w lies rather close to $z_{v^c_{+1}}$. Quantitatively speaking we have

$$\rho(z_w, \omega) \leq K^5 \min\{\rho(z_{v^c_{+1}}, z_w), \rho(z_{v^c_{+1}}, \omega)\} \;\forall z_w, z_{v^c_{+1}}. \tag{6.18}$$

Now since v is regular, there is a $z_{v^c_{+1}}$ with $\rho(z_{v^c_{+1}}, \omega) \leq Kr^{l(v)}$. Then by (6.18), $\rho(z_w, \omega) \leq K^6 r^{l(v)}$.

On the other hand $B(w)$ must not contain ω, so $\rho(z_w, \omega) \geq Kr^{l(w)}$. It thus follows that $l(w)\dot{\geq}l(v)$ up to a uniform error, or in words that $B(w)$ is smaller than $B(v)$ up to a uniform factor.

But then

$$\sup \rho(z_v, z_w) \asymp \sup \rho(z_v, \omega).$$

In addition,

$$\sup \rho(z_v, \omega) \asymp \sup \rho(z_v, w),$$

since $B(w)$ is contained within the ball of radius $K^6 r^{l(v)}$ around ω. This proves (6.17).

It remains to prove the lemma for v, w both not horizontally connected to nor on the ray.

We start again with (6.12),

$$\frac{\mathrm{diam}_{\rho'}(B(F(v)))\mathrm{diam}_{\rho'}(B(F(w)))}{\sup \rho'(z_{F(v)}, z_{F(w)})^2} \asymp \frac{\mathrm{diam}_\rho(B(v))\mathrm{diam}_\rho(B(w))}{\sup \rho(z_v, z_w)^2}.$$

Since v is not connected to the ray, we get, just as in the case above

$$\rho'(z,v) = \frac{\rho(z,v)}{\rho(z,\omega)\rho(v,\omega)} \asymp_K \frac{\rho(z,v)}{\rho(v,\omega)^2} \ \forall z \in B(v)$$

and thus

$$\mathrm{diam}_{\rho'}(B(v)) \asymp \frac{\mathrm{diam}_\rho(B(v))}{\rho(v,\omega)^2}.$$

The same estimate also holds for $\mathrm{diam}_{\rho'}(B(w))$. (6.12) becomes

$$\frac{\mathrm{diam}_\rho(B(v))\mathrm{diam}_\rho(B(w))}{\rho(v,\omega)^2\rho(w,\omega)^2 \sup \rho'(z_v,z_w)^2} \asymp \frac{\mathrm{diam}_\rho(B(v))\mathrm{diam}_\rho(B(w))}{\sup \rho(z_v,z_w)^2},$$

which is equivalent to

$$\sup \rho'(z_v,z_w)\rho(v,\omega)\rho(w,\omega) \asymp \sup \rho(z_v,z_w).$$

This follows if we can show that

$$\rho'(z_v,z_w)\rho(v,\omega)\rho(w,\omega) \asymp_C \rho(z_v,z_w) \ \forall z_v, z_w \tag{6.19}$$

for some uniform constant C. But (6.19) is equivalent to

$$\frac{\rho(z_v,z_w)}{\rho(z_v,\omega)\rho(z_w,\omega)}\rho(v,\omega)\rho(w,\omega) \asymp_C \rho(z_v,z_w).$$

It thus suffices to show

$$\rho(z_v,\omega) \asymp \rho(v,\omega)$$
$$\rho(z_w,\omega) \asymp \rho(w,\omega),$$

and these estimates hold because $\rho(z_v,\omega) > Kr^{l(v)}$, so in

$$\{\rho(z_v,\omega), \rho(v,\omega), \rho(v,z_v)\}$$

the minimum will always, that is, for all $z_v \in B(v)$, be $\rho(v,z_v)$, thus $\rho(z_v,\omega) \asymp_K \rho(v,\omega)$. The same holds for w. The lemma follows. □

Corollary 98. *Let* v, w arbitrary regular *vertices. Then* $|F(v)F(w)| \doteq |vw|$.

Proof. This follows from Lemma 97. □

Lemma 99. *Let v, w be the top and lower ends respectively of a singular segment in* $\mathrm{Hyp}\,(Z,\rho)$*. Then $F(v), F(w)$ are uniformly close to the ends of a singular segment in* $\mathrm{Hyp}\,(Z,\rho')$ *of roughly the same length.*

Proof. First assume that v is not on the ray $o\omega$ and not horizontally connected to it. Consider $z_0, z_1 \in B(v)$ and $z_2 \in B(v)^c$. Then

$$\frac{\rho'(z_0,z_1)}{\rho'(z_0,z_2)} = \frac{\rho(z_0,z_1)\rho(z_2,\omega)}{\rho(z_1,\omega)\rho(z_0,z_2)}.$$

This cannot be (much) larger than $\rho(z_0,z_1)/\rho(z_0,z_2)$, which implies that $F(v)$ is the top end of a singular segment of length $\dot{\geq}|vw|$.

If we can prove that $\ell(F(w)) < \ell(F(v))$, then a geodesic joining $F(w)$ to $F(v)$ will reach $F(v)$ from below, thus has to go through the singular segment. Since $|F(v)F(w)| \doteq |vw|$ by Lemma 97, the lemma follows.

Now if w is neither on the ray $o\omega$ nor horizontally connected to it, then

$$\frac{\rho'(v,z_v)}{\rho'(w,z_w)} = \frac{\rho(v,z_v)}{\rho(w,z_w)} \cdot \frac{\rho(w,\omega)\rho(z_w,\omega)}{\rho(z_v,\omega)\rho(v,\omega)} \asymp_{K^2} \frac{\rho(v,z_v)}{\rho(w,z_w)},$$

whence $l(w) \dot{\leq} l(v)$. Similar estimates hold in case w is connected to or on the ray $o\omega$, that is, the ρ'-diameter of $B(w_{+1})^c$ is much larger than that of $B(v)$, where , again, the notation $B(w_{+1})$ means the ball associated to $(\pi(w), l(w)+1)$, i.e. $B_{Kr^{l(w)+1}}(\pi(w))$. This proves the lemma in case v is not horizontally connected to, nor on the ray.

Finally, if $\pi(v) = \omega$, then also $\pi(w) = \omega$ or $F(w) = F(\tilde{w})$ with $\pi(\tilde{w}) = \omega$ ($\tilde{w}$ being a horizontal neighbor to w on the ray). It follows immediately by definition of ρ' that there is a singular segment of roughly the same length between $F(v)$ and $F(w)$ (as long as $w \neq o$, but in this case simply apply the definition of F). □

Now we show that a root of a singular ray in $\mathrm{Hyp}\,(Z,\rho)$ is mapped uniformly close to the root of a singular ray in $\mathrm{Hyp}\,(Z,\rho')$.

Lemma 100. *There is a one-to-one correspondence between singular rays in* $\mathrm{Hyp}\,(Z,\rho)$ *and* $\mathrm{Hyp}\,(Z,\rho')$ *and a root of a singular ray in* $\mathrm{Hyp}\,(Z,\rho)$ *is mapped uniformly close to a (hence, any) root of the associated singular ray in* $\mathrm{Hyp}\,(Z,\rho')$*, with the exception of a singular ray in* $\mathrm{Hyp}\,(Z,\rho)$ *going to* ω*, which is mapped to a singular ray "downwards" to* ∞ *in* $\mathrm{Hyp}\,(Z,\rho')$*.*

Proof. That there is a one-to-one correspondence is clear because every singular ray corresponds to an isolated point in the boundary, and $\mathrm{id}|_{Z\setminus\{\omega\}}$ is a homeomorphism onto its image, so maps isolated points to isolated points, and if there is a singular ray to ω then $(Z \setminus \{\omega\}, \rho')$ is bounded, so there will be an associated singular ray descending to ∞ in $\mathrm{Hyp}\,(Z,\rho')$. We just need to argue that the root of a ray associated to z in $\mathrm{Hyp}\,(Z,\rho)$ is mapped close to the root of the ray associated to z in $\mathrm{Hyp}\,(Z,\rho')$. Assume first that if v is a root of the ray associated to z, then either v is not connected to nor on the ray $o\omega$, or if it is on the ray, then it is at least two levels above o.

Now note $B(F(v))$ contains z by definition. It therefore suffices to show that the level of $F(v)$ is roughly the same as that of the root q of the ray associated to z in $\mathrm{Hyp}\,(Z,\rho')$. Now if v is not connected to nor on the ray $o\omega$, then $\mathrm{diam}_{\rho'}(B(v)) \asymp \mathrm{diam}_{\rho}(B(v))/\rho(v,\omega)^2$ and similarly $\inf_{z'\neq z} \rho'(z,z') \asymp \inf \rho(z,z')/\rho(v,\omega)^2$, hence the levels of q and $F(v)$ agree up to uniform error. If on the other hand v is centered at ω

$$\inf_{z'} \rho'(z,z') = \inf \frac{\rho(z,z')}{\rho(z,\omega)\rho(z',\omega)} \geq \frac{1}{\min\{\rho(z,\omega), \rho(z',\omega)\}},$$

and since v is at least two levels above the root, there exists z' such that $\rho(z',\omega) > K\rho(z,\omega)$, i.e. $\rho(z,\omega) \asymp_K \rho(z,z')$. It follows that $\inf_{z'} \rho'(z,z') \asymp 1/\rho(z,\omega)$. The same argument yields that $\mathrm{diam}_{\rho'}(B(v_{+1}))^c \asymp 1/\rho(z,\omega)$.

For the exceptional cases where the root v is equal to o, to $(\pi(o), \ell(o)+1)$, or horizontally connected to the latter, one shows with similar arguments that if R_1, R_2 are two singular rays with the *same* exceptional root v, then the roots q_1, q_2 of the associated singular rays in $\mathrm{Hyp}\,(Z, \rho')$ are uniformly close to each other. Since there are only 3 types of exceptional roots, it follows that the distance between the image $F(v)$ of the root and the root q of the ρ'-ray associated to z is uniformly bounded, $|F(v)q| \doteq 0$. $\square$

It follows readily that a roughly isometric map between geodesic spaces which induces a surjective boundary map is a rough isometry. The only thing left to show in the proof of Theorem 92, then, is that $\partial_\infty F = id_Z$. That a sequence converging to ω is mapped to $\infty \in (Z, \rho')$ is clear by definition of F. If $\{v_i\}$ is a sequence converging to infinity, say $\{v_i\} \in z$, $z \neq \omega$, we may suppose by [BS07] Lemma 6.3.2 that the v_i form a radial geodesic in $\mathrm{Hyp}\,(Z, \rho)$. Since F is a rough isometry, $\{F(v_i)\}$ converges to a point $z' \in (Z, \rho')$. But $F(v_i)$ is the smallest ρ'-ball containing $B_\rho(v_i)$, which contains z. Since $\rho(\pi(v_i), \pi(F(v_i))) \overset{i\to\infty}{\to} 0$ (the levels of $F(v_i)$ go to infinity), we have $\rho'(\pi(F(v_i)), z) \overset{i\to\infty}{\to} 0$, i.e. $\partial_\infty F(z) = z \ \forall z \in Z$. This completes the proof of Theorem 92. $\square$

6.4 Extension of P-QM Maps

In this section we prove

Theorem 101. *Let $f : (Z, \rho) \to (Z', \rho')$ a power quasimoebius homeomorphism between complete quasimetric spaces. Then there exists a power quasi-isometry $F : \mathrm{Hyp}\,(Z) \to \mathrm{Hyp}\,(Z')$ with $\partial_\infty F = f$.*

The idea of the proof is to factor f as a composition of inversions and a P-QS map. We follow 3.15 of [Väi85], where this factorization is explained in the metric setting.

Lemma 102 (Cf. [Väi85] Thm 2.1)**.** *Let $(X, \rho), (Y, \rho')$ be bounded quasimetric spaces and $f : X \to Y$ be θ-quasimoebius. Let $z_1, z_2, z_3 \in X$ and $\lambda > 0$ be such that $\rho(z_i, z_j) \geq \mathrm{diam}(X)/\lambda$ and $\rho'(f(z_i), f(z_j)) \geq \mathrm{diam}(Y)/\lambda$ when $i \neq j$. Then there is a homeomorphism $\mu : [0, \infty) \to [0, \infty)$, depending only on θ and λ and the quasimetric constant K of X, such that*

$$\rho'(f(x), f(y)) \leq \mathrm{diam}(Y)\mu(\rho(x, y)/\mathrm{diam}(X)).$$

Moreover, if θ is of power type, then μ can also be taken of power type.

Proof. In analogy to the proof of [Väi85] Thm 2.1, consider the cases

1. $\rho(x, z_1) < 1/K$ and $\rho(x, y) < 1/K^2$,
2. $\rho(x, y) \geq 1/K^2$,
3. $\rho(x, z_1) \geq 1/K$,

and follow the same arguments as in that proof, replacing any occurrence of the usual triangle inequality by the quasimetric version. Although not mentioned in [Väi85], the fact that μ inherits power type is implied by the proof, cf. Appendix B. $\square$

Lemma 103 (Cf. [Väi85] Thm 3.12)**.** *Suppose $f : X \to Y$ is a QM map between bounded quasimetric spaces. Then f is QS. If f is P-QM, then f is P-QS.*

Proof. Also here the proof of [Väi85] can be "quasified". Set $r_0 := \mu^{-1}(\mu^{-1}(1/K^2))$ and $r_1 := \min\{1/(K^2 t), r_0/(Kt), r_0/K\}$. Then consider the cases

1. $r \geq r_1$,
2. $r < r_1$ and $\rho'(f(x), f(z_1)) \geq \mu^{-1}(1/K^2)$,
3. $r < r_1$ and $\rho'(f(x), f(z_1)) < \mu^{-1}(1/K^2)$,

and follow analogous arguments to [Väi85]. Careful inspection of that proof also yields the inheritance of power type, cf. Appendix B. □

Lemma 104 below is the quasimetric analog of another theorem in [Väi85].

Lemma 104 (Compare [Väi85] Thm. 1.10)**.** *Every (complete) quasimetric space (Z, ρ) is Moebius-equivalent to a (complete) bounded quasimetric space.*

Proof. Fix a $z_0 \in Z$, consider the set $Y = Z \cup \{\xi\}$, and equip it with the quasimetric $\widetilde{\rho}$ defined as $\widetilde{\rho}|_{Z\times Z} = \rho$ and $\widetilde{\rho}(\xi, z) = 1 + \rho(z_0, z)$. Then the canonical embedding $\iota : Z \hookrightarrow Z \cup \{\xi\}$ is an isometry. Invert $\widetilde{\rho}$ in ξ. □

We now have all the tools to prove the theorem.

Proof of Thm. 101. Let $\iota_i : (Z_i, \rho_i) \hookrightarrow Y_i$, $i = 1, 2$, be the embeddings as in the proof above. Let $z_i \in Z_i$ be fixed and denote by $u_i : Y_i \to Y_i$ the inversion in z_i as in the proof above. Then $v_i := u_i \circ \iota_i$ are Moebius homeomorphisms from (Z_i, ρ_i) onto their bounded images in Y_i.

Then $g := (u_2 \circ \iota_2) \circ f \circ (u_1 \circ \iota_1)|^{-1}_{u_1 \circ \iota_1(Z_1)}$ is a PQ-Moebius homeomorphism between two bounded quasimetric spaces, thus it is PQ-symmetric by Lemma 103.

Thus f decomposes as $f = (u_2 \circ \iota_2)^{-1} \circ g \circ (u_1 \circ \iota_1)$. The claim follows with Theorems 92 and 78. Note that $\partial_\infty(F \circ G) = \partial_\infty F \circ \partial_\infty G$, purely by definition of the boundary maps. □

6.5 Summary of Extension Theorems

The following theorem summarizes what we have proved so far.

Theorem 105. *Let X, X' visual roughly geodesic hyperbolic metric spaces. The following are equivalent.*

I) X and X' are roughly isometric.

II) There is a map $F : X \to X'$ and a $D \geq 0$ such that for all quadruples $Q \subset X$

$$\mathrm{cd}(Q) - D \leq \mathrm{cd}(F(Q)) \leq \mathrm{cd}(Q) + D.$$

III) For any $a > 1$ there is a bilipschitz-quasimoebius homeomorphism $f : \partial^a_\infty X \to \partial^a_\infty X'$.

Also the following are equivalent.

i) X and X' are quasi-isometric.

ii) X and X' are power quasi-isometric.

iii) For any $a, a' > 1$, $\partial_\infty^a X$ is power quasimoebius-equivalent to $\partial_\infty^{a'} X'$.

Proof. As mentioned in the Introduction, the implications $II) \Rightarrow I)$, $I) \Rightarrow II)$ and $ii) \Rightarrow i)$ are all trivial.

$I) \Rightarrow III)$: It is clear that if X, X' are roughly isometric, then $\partial_\infty^{a,o} X, \partial_\infty^{a,o'} X'$ are bilipschitz equivalent for any $o \in X$, $o' \in X'$. Also, a bilipschitz map is obviously bilipschitz-quasimoebius. Remains to show that $\partial_\infty^{a,o} X$ and $\partial_\infty^{a,b} X$, with $b \in B(\omega)$ for some $\omega \in \partial_\infty X$, are bilipschitz-quasimoebius equivalent. But if we take the distinguished Busemann function $b_{\omega,o}(x) := (\omega|o)_x - (\omega|x)_o$, it by definition induces the inverted quasimetric $\rho'(\cdot,\cdot) = a^{-(\cdot|\cdot)_{b_{\omega,o}}}$ to $\rho(\cdot,\cdot) = a^{-(\cdot|\cdot)_o}$ on $\partial_\infty X$,

$$\rho'(\xi,\eta) = \frac{\rho(\xi,\eta)}{\rho(\xi,\omega)\rho(\eta,\omega)},$$

so that $\partial_\infty^{a,o} X$ and $\partial_\infty^{a,b_{\omega,o}} X$ are Moebius-equivalent (no quasi). Now, by definition (cf. [BS07] §3.1), any $b \in \mathcal{B}(\omega)$ satisfies $b \doteq b_{\omega,o} - C$, for some C, and thus $\partial_\infty^{a,b} X$ and $\partial_\infty^{a,b_{\omega,o}} X$ are bilipschitz-quasimoebius equivalent.

$III) \Rightarrow I)$: By Thm. 92 and Rem. 93, we may pre- and post-compose with inversions if necessary to reduce this to the bounded case $\partial_\infty^{a,o} X$, $\partial_\infty^{a,o'} X'$. By Lemma 106, f is bilipschitz. The claim now follows from the bilipschitz extension Theorem 75.

$i) \Rightarrow ii)$: This is Thm. 4.4.1 of [BS07].

$i) \Rightarrow iii)$: This is Prop. 5.2.10 of [BS07].

$iii) \Rightarrow i)$: This follows from the third statement of Thm. 101 and the fact, due essentially to the stability of quasi-geodesics, that a quasi-isometric map between visual roughly geodesic spaces which induces a bijective boundary map is necessarily a quasi-isometry (cf. [BS07] Lemma 7.3.12). ☐

Lemma 106. *If $f : Z \to Z'$ is a bilipschitz-QS map between quasimetric spaces, then f is bilipschitz. If Z, Z' are* bounded *and f is bilipschiz-QM, then f is bilipschitz.*

Proof. Let f bilipschitz-QS, i.e. η-QS with $\eta(t) = \mu t$ for some constant μ. Fix $a, b \in Z$, $a \neq b$, and set $\Delta := |a'b'|/|ab|$, where $'$ denotes images under f. For $x \neq y$ in Z, $y \neq a$, $x \notin \{a, b\}$, we have

$$\frac{|x'y'|}{|xy|} \asymp_\mu \frac{|x'a'|}{|xa|} \asymp_\mu \frac{|a'b'|}{|ab|} = \Delta,$$

whence $|x'y'| \asymp_{\mu^2 \Delta} |xy|$. The exceptional cases $y = a$, $x = a$ or $x = b$ are treated the same way.

The second statement is an immediate consequence of the fact that that a bilipschitz-QM map between bounded quasimetric spaces is bilipschitz-QS, a result essentially due to Väisälä ([Väi85] Thm. 3.12). We give a detailed proof of this fact in Appendix B. ☐

6.6 Uniqueness of the Extension

Here we prove that power quasimoebius boundary maps extend to unique (up to a rough isometry) PQ-isometries between associated visual geodesic hyperbolic spaces if and only if the boundaries are uniformly perfect.

Note that the "only if" part is clear; in any visual geodesic space whose boundary is not uniformly perfect, it is easy to define a quasi-isometry which is not in the same rough mapping class as the identity map. We prove here the "if" direction.

First some elementary lemmata.

Lemma 107. *In any geodesic Gromov hyperbolic metric space, asymptotic rays have uniformly close* (2δ) *terminal segments.*

Proof. For a sketch of the situation, cf. Fig. 6.4. Let γ, γ' two rays with $\gamma(\infty) = \gamma'(\infty)$. Denote $o = \gamma(0), o' = \gamma'(0)$. We want to show that for t sufficiently large, $\mathrm{dist}(\gamma'(t), \gamma) \leq 2\delta$. Let $t \geq (o|\xi)_{o'} + \delta$. Since $(\gamma'(m)|\gamma(n))_{o'} \to \infty$, it is easy to find an n such that $\gamma'(t)$ is δ-close to some $x \in o'\gamma(n)$. But then $|o'x| \geq t - \delta \geq (o|\xi)_{o'}$, whence, considering the triangle $oo'\gamma(n)$, x is δ-close to the side $0\gamma(n)$. □

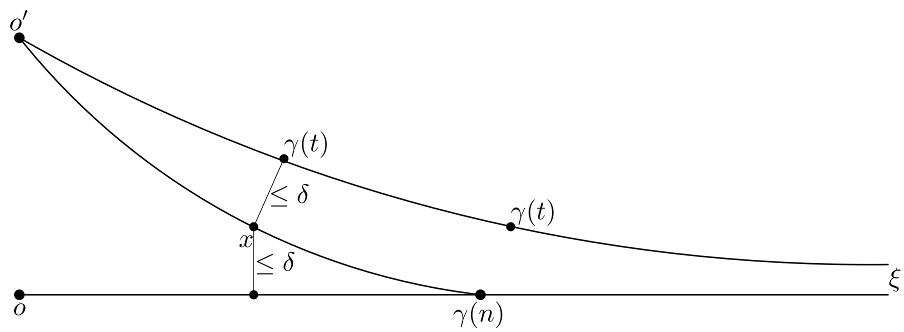

Figure 6.4: Asymptotic rays are uniformly close

Lemma 108. *Let X be Gromov hyperbolic such that for every $x \in X$, $\xi \in \partial_\infty X$ there exists a ray $x\xi$. Let $\gamma : \mathbb{R}_{\geq 0} \to X$ be a geodesic ray and $F : X \to X$ a quasi-isometric map with $\partial_\infty F = \mathrm{id}_{\partial_\infty X}$. Then $F(\gamma)$ has a terminal segment that is uniformly close to γ (error depending on F, δ).*

Proof. $F(\gamma)$ is a quasigeodesic ray with the same end as γ. Moreover, it is uniformly close to any geodesic segment between any two of its points. Look at $F(\gamma(0))$ and a ray η from $F(\gamma(0))$ to $\gamma(\infty)$, cf. Fig. 6.5. By Lemma 107, a terminal segment of η is uniformly close to γ. We want to know why $F(\gamma)$ is uniformly close to η. Now $(F(\gamma(t))|\eta(t))_{F(\gamma(0))} \to \infty$, since both tend toward ξ. Since the initial parts of $F(\gamma)$ are all uniformly close to geodesics $F(\gamma(0))F(\gamma(t))$, and since these are uniformly close to initial segments of η, initial segments of $F(\gamma)$ must also be uniformly close to η. These initial segments can be taken arbitrarily large, and since η eventually gets uniformly close to γ, so must $F(\gamma)$. □

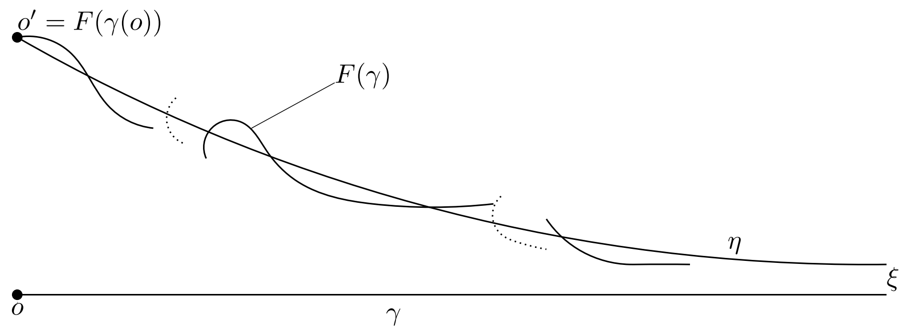

Figure 6.5: Image of a geodesic is uniformly close

Corollary 109. *Let X be geodesic and Gromov hyperbolic such that for every $x \in X$, $\xi \in \partial_\infty X$ there exists a ray $x\xi$. Let $\gamma : \mathbb{R} \to X$ be a bi-infinite geodesic and $F : X \to X$ a quasi-isometric map with $\partial_\infty F = \mathrm{id}_{\partial_\infty X}$ ($\partial_\infty F(\gamma(\pm\infty)) = \gamma(\pm\infty)$ is enough). Then $F(\gamma)$ is uniformly close to γ (error depending on F, δ).*

Proof. By Lemma 107, there are terminal segments of γ and $F(\gamma)$ that are uniformly close, say for all $t > T$, $F(\gamma(\pm t))$ is uniformly close to γ. A geodesic between $F(\gamma(t))$ and $F(\gamma(-t))$ are uniformly close to $F(\gamma)$ (restricted to $[-t, t]$), but it is also clearly uniformly close to γ (restricted to an appropriate interval).Thererfore $F(\gamma)$ is also uniformly close to γ in between the terminal segments. $\square$

For illustrative purposes, we now first prove the Uniqueness Theorem for the hyperbolic plane.

Theorem 110. *If $F : \mathbb{H}^2 \to \mathbb{H}^2$ is a quasi-isometry with $\partial_\infty F = \mathrm{id}_{S^1}$. Then F is roughly the identity.*

Proof. Let $x \in \mathbb{H}^2$. Pick two bi-infinite geodesics γ_1, γ_2 which intersect in x at a right angle. By Lemma 109, $F(x)$ is in a uniform C-neighborhood of both γ_1 and γ_2. Consider $\mathrm{p}_1 : \mathbb{H}^2 \to \gamma_1$ the orthogonal projection onto γ_1, cf. Fig. 6.6. This projection is distance non-increasing (in fact, it decreases distances). We have

$$\begin{aligned} |xF(x)| &\leq |x\,\mathrm{p}_1(F(x))| + |\,\mathrm{p}_1(F(x))F(x)| \\ &\leq \mathrm{dist}(F(x), \gamma_2) + \mathrm{dist}(F(x), \gamma_1) \leq 2C.^1 \end{aligned}$$

$\square$

Now we adapt this proof to a general setting.

Theorem 111. *Let X a visual geodesic Gromov hyperbolic space and $\partial_\infty X$ its boundary at infinity equipped with any arbitrary visual metric. If $\partial_\infty X$ is uniformly perfect, then any PQ-isometry of X to itself that induces the identity in the boundary is roughly the identity. More precisely, if $F : X \to X$ is P-QI with $\partial_\infty F = \mathrm{id}_{\partial_\infty X}$, then there is a constant C, depending on F and the hyperbolicity constant δ of X, such that $F(x) \doteq_C x$, $\forall x \in X$.*

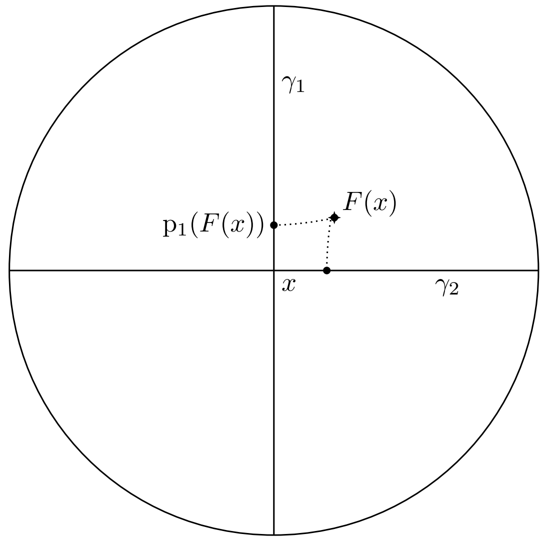

Figure 6.6: Situation in $\mathbb{H}^2$

Remark 112. *For rough isometries the assumptions that the space is geodesic and its boundary uniformly perfect can be dropped. Namely, if a rough isometry on a* visual *hyperbolic space induces the identity in the boundary, then it is roughly the identity.*

The following is an immediate consequence of Thm. 111.

Corollary 113. *Let X, Y visual geodesic Gromov hyperbolic, $\partial_\infty X$ uniformly perfect and $f : \partial_\infty X \to \partial_\infty Y$ a P-QM homeomorphism.*

Then there exists a unique rough mapping class of power quasi-isometric maps $F : X \to Y$ with $\partial_\infty F = f$. In other words, for any $\tilde{F} : X \to Y$ with $\partial_\infty \tilde{F} = f$, there exists a C such that $|F(x)\tilde{F}(x)| \leq C$ for all $x \in X$.

Proof of Thm. 111. Since X is visual and geodesic, it is roughly isometric to a hyperbolic approximation of its boundary. We thus treat X as such an approximation. In fact, we may w.l.o.g. assume that the boundary is equipped with an *extended* quasimetric. The fact that the approximation then has no root is convenient in the proof.

Let $x \in X$ arbitrary. Uniform perfection of the boundary means that singular segments in X are of uniformly bounded length D, say. This means we can find two bi-infinite geodesics γ_1, γ_2 which pass through x and whose intersection consists of a singular segment of length at most D, the upper and lower ends of which we denote by A and B respectively, cf. Fig. 6.7. Let $\mathrm{p_i} : X \to \gamma_i$ the nearest-point-projection of X onto γ_i, $i = 1, 2$. These projections are well-defined up to error 1, as some $x \in X$ may have two distinct closest points on γ_i, but these are then at distance 1 of each other. Then $\mathrm{p}_1(\gamma_2) \doteq_1 AB$, where the rough equality means that the two sets have Hausdorff distance no more than 1. Now in general $\mathrm{p}_1(\mathrm{p}_2(F(x))$ will be some point on the segment AB, and since this segment has uniformly bounded length, we may assume that $\mathrm{p}_1(\mathrm{p}_2(F(x)) = A$.

Then

$$|\,\mathrm{p}_2(F(x))A| \le |\,\mathrm{p}_2(F(x))\,\mathrm{p}_1(F(x))| \le |\,\mathrm{p}_2(F(x))F(x)| + |F(x)\,\mathrm{p}_1(F(x))| \le 2C.$$

Consequently

$$\begin{aligned}|x(F(x))| &\le |x\,\mathrm{p}_1(F(x))| + |\,\mathrm{p}_1(F(x))F(x)| \le |xA| + |A\,\mathrm{p}_1(F(x))| + C \\ &\le D + |A\,\mathrm{p}_2(F(x))| + |\,\mathrm{p}_2(F(x))F(x)| + |F(x)\,\mathrm{p}_1(F(x))| + C. \\ &\le D + 5C.\end{aligned}$$

□

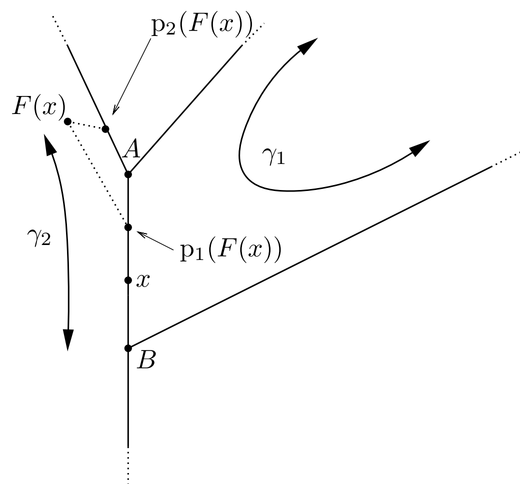

Figure 6.7: Setting for the proof of Thm. 111

Chapter 7

∂_∞ and Hyp as Functors

The results of Ch. 6 suggest that ∂_∞ and Hyp can be regarded as functors between appropriate categories, and indeed there are various ways to make this precise. Some steps in this direction were undertaken in [BS00]. For example, for every $a > 1$ one could consider ∂_∞^a as a functor from the category of "pointed" Gromov hyperbolic spaces (we have to specify base points or Busemann functions!) to the category of complete quasimetric spaces, and similarly Hyp_r as a functor in the opposite direction. The problem is that these functors do not have many nice properties. Generally, ∂_∞ will not be faithful and Hyp will not be full. In particular, $\partial_\infty \circ \mathrm{Hyp}$ and $\mathrm{Hyp} \circ \partial_\infty$ are not the identity functors. Even worse, they are not even the identity map on objects. But the intuition is that they should be in some sense inverse to each other. In order to make this precise, one should not look at categories where the objects are individual spaces, but rather appropriate *structures* on spaces. This way of thinking is also more in line with the philosophy of Gromov hyperbolic spaces, where one is inclined to consider two spaces the same if one is just a bounded distortion of the other.

Definition 114. *We call two quasimetrics ρ_1, ρ_2 on a set Z* bilipschitz-quasimoebius equivalent *if the identity map* $\mathrm{id} : (Z, \rho_1) \to (Z, \rho_2)$ *is a bilipschitz-quasimoebius homeomorphism.*

A bilipschitz-quasimoebius structure *(BL-QM structure) is an equivalence class of this relation. Such a structure is called* complete *if one, and hence any, of its members is a complete quasimetric space.*

Analogously we can define Hölder structures *on a set as the class of all quasimetrics on this set such that the identity map between any two of them satisfies a two-sided Hölder bound, i.e.*

$$\frac{1}{C}\rho_1^\alpha(z_1, z_2) \le \rho_2(z_1, z_2) \le C\rho_1^\alpha(z_1, z_2)$$

for some C, α.

Thus for every $a > 1$, ∂_∞^a determines for every Gromov hyperbolic space X a unique bilipschitz-quasimoebius structure on the set $\partial_\infty X$ and the symbol $\partial_\infty X$ can be interpreted as defining for every Gromov hyperbolic space X a unique Hölder structure on the set $\partial_\infty X$.

These define appropriate structures for our categories of boundaries. Analogously, we define the following structures for Gromov hyperbolic spaces.

Recall that two metric spaces X, Y are called roughly isometric if there is a rough isometry $F : X \to Y$. This is an equivalence relation among metric spaces. Similarly, call two maps $F : X \to Y$, $F' : X' \to Y'$ *equivalent up to rough isometry* if there are rough isometries $R_X : X \to X'$ and $R_Y : Y' \to Y$ such that $F \doteq R_Y \circ F' \circ R_X$, cf. Fig. 7.1 below. This is an equivalence relation among maps between metric spaces.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "R_X"'] & Y \\ X' \arrow[r, "F'"'] & Y' \arrow[u, "R_Y"'] \end{tikzcd}$$

Figure 7.1: Equivalence of maps.

Replacing rough isometries with rough similarities, we obtain analogous notions for equivalence of maps up to *rough similarities*.

Definition 115. *A* rough isometry class of metric spaces *is an equivalence class of the relation between metric spaces to be roughly isometric to one another.*

A rough isometry class of maps between metric spaces *is an equivalence class of the relation between maps of metric spaces to be equivalent up to rough isometries.*

A rough similarity class of metric spaces *is an equivalence class of the relation between metric spaces to be roughly similar to one another.*

A rough similarity class of maps between metric spaces *is an equivalence class of the relation between maps of metric spaces to be equivalent up to rough similarities.*

Recall ([Mey09] Def. 5.1) that a metric space X is called *uniformly equilateral* if there exist two numbers $S_0 > 0,\ \lambda > 0$, such that for every $w \in X$ and every $S \geq S_0$ the ball $B_S(w) \subset X$ contains three points x, y, z with

$$(x|y)_z,\ (y|z)_x\ (x|z)_y \geq \lambda S.$$

Definition 116. *Denote by $\mathcal{C}_{[P-QI]_{r.i.}}$ the category with objects rough isometry classes of Gromov hyperbolic spaces and with morphisms rough isometry classes of P-QI maps.*

Denote by $\mathcal{D}_{[P-QM]_{BL-QM}}$ the category with objects complete BL-QM structures of quasimetric spaces and morphisms BL-QM classes of P-QM maps.

Furthermore, superscripts $^{u.e.}$, ^{v}, $^{r.g.}$ and $^{u.p.}$ stand for the corresponding subcategories where only objects are considered that are uniformly equilateral, visual, roughly geodesic and uniformly perfect, respectively. This is well-defined as these properties are invariant under members of a fixed class. For example $\mathcal{C}^{v,r.g.}_{[P-QI]_{r.i.}}$ would be the category with objects rough isometry classes of visual roughly geodesic *Gromov hyperbolic spaces as objects and rough isometry classes* *of P-QI maps as morphisms. Similarly, $\mathcal{D}^{u.p.}_{[P-QM]_{BL-QM}}$ are the complete* uniformly perfect *BL-QM structures and BL-QM classes of P-QM maps between them.*

These categories turn out to be suitable for the functors ∂^a_∞ and Hyp_r. The results from Ch. 6 imply the following statements.

Proposition 117 (∂^a_∞ and Hyp_r as functors)**.** *For any $a > 1$ and $r \in (0, 1)$ we have:*

$$
\begin{array}{rcll}
\partial^a_\infty : \mathcal{C}_{[P-QI]_{r.i.}} & \longrightarrow & \mathcal{D}_{[P-QM]_{BL-QM}} & \textit{is a functor.} \\
\partial^a_\infty : \mathcal{C}^{v,r.g.}_{[P-QI]_{r.i.}} & \longrightarrow & \mathcal{D}_{[P-QM]_{BL-QM}} & \textit{is } \text{full.} \\
\partial^a_\infty : \mathcal{C}^{u.e.,v}_{[P-QI]_{r.i.}} & \longrightarrow & \mathcal{D}^{u.p.}_{[P-QM]_{BL-QM}} & \textit{is } \text{faithful.} \\
\partial^a_\infty : \mathcal{C}^{u.e.,v,r.g.}_{[P-QI]_{r.i.}} & \longrightarrow & \mathcal{D}^{u.p.}_{[P-QM]_{BL-QM}} & \textit{is } \text{full and faithful.} \\
\mathrm{Hyp}_r : \mathcal{D}_{[P-QM]_{BL-QM}} & \longrightarrow & \mathcal{C}^{v,r.g.}_{[P-QI]_{r.i.}} & \textit{is } \text{faithful.} \\
\mathrm{Hyp}_r : \mathcal{D}^{u.p.}_{[P-QM]_{BL-QM}} & \longrightarrow & \mathcal{C}^{u.e.,v,r.g.}_{[P-QI]_{r.i.}} & \textit{is } \text{full and faithful.}
\end{array}
$$

Moreover, $\mathrm{Hyp}_{1/a}$ *and ∂^a_∞ are mutually inverse functors between $\mathcal{C}^{u.e.,v,r.g.}_{[P-QI]_{r.i.}}$ and $\mathcal{D}^{u.p.}_{[P-QM]_{BL-QM}}$.* □

Remark 118. *∂^a_∞ and* $\mathrm{Hyp}_{1/a}$ *are in fact bijective on* objects *as soon as we consider visual geodesic Gromov hyperbolic spaces on the one hand and arbitrary complete quasimetric spaces on the other hand. But if one requires also bijectivity on* morphisms*, it becomes necessary to restrict to hyperbolic spaces which are in addition uniformly equilateral and to uniformly perfect boundaries.*

Since $\mathrm{Hyp}_r Z$ and $\mathrm{Hyp}_{r'} Z$ are basically just rescalings of each other for different r, r' and $\partial^a_\infty X$ is Hölder equivalent to $\partial^{a'}_\infty X$, one might want to define these as the same objects.

Definition 119. *Denote by $\mathcal{C}_{[P-QI]_{r.s.}}$ the category with objects rough similarity classes of Gromov hyperbolic spaces and with morphisms rough similarity classes of P-QI maps.*

Denote by $\mathcal{D}_{[P-QM]_H}$ the category with objects complete Hölder structures of quasimetric spaces and morphisms Hölder classes of P-QM maps.

The same rules for superscripts as in Def. 116 apply.

We can define ∂_∞ and Hyp as functors between these categories.

Definition 120. *For $[X]$ a rough similarity class of Gromov hyperbolic spaces, define $\partial_\infty[X]$ to be the Hölder class of complete quasimetric spaces that contains $\partial^a_\infty X$ for some, and hence any, $a > 1$.*

For $[Z]$ a Hölder class of complete quasimetric spaces, define $\mathrm{Hyp}\,[Z]$ *to be the rough similarity class of metric spaces that contains* $\mathrm{Hyp}_r Z$ *for some, and hence any, $r \in (0, 1)$.*

$\partial_\infty[F]$ and $\mathrm{Hyp}\,[f]$*, for morphisms $[F]$ and $[f]$ of $\mathcal{C}$ and $\mathcal{D}$ respectively, are defined analogously.*

Proposition 121 (∂_∞ and Hyp as functors)**.**

$$\begin{array}{rll}
\partial_\infty : \mathcal{C}_{[P-QI]_{r.s.}} \longrightarrow \mathcal{D}_{[P-QM]_H} & & \textit{is a functor.} \\
\partial_\infty : \mathcal{C}^{v,r.g.}_{[P-QI]_{r.s.}} \longrightarrow \mathcal{D}_{[P-QM]_H} & & \textit{is } \text{full.} \\
\partial_\infty : \mathcal{C}^{u.e.,v}_{[P-QI]_{r.s.}} \longrightarrow \mathcal{D}^{u.p.}_{[P-QM]_H} & & \textit{is } \text{faithful.} \\
\partial_\infty : \mathcal{C}^{u.e.,v,r.g.}_{[P-QI]_{r.s.}} \longrightarrow \mathcal{D}^{u.p.}_{[P-QM]_H} & & \textit{is } \text{full and faithful.} \\
\mathrm{Hyp} : \mathcal{D}_{[P-QM]_H} \longrightarrow \mathcal{C}^{v,r.g.}_{[P-QI]_{r.s.}} & & \textit{is } \text{faithful.} \\
\mathrm{Hyp} : \mathcal{D}^{u.p.}_{[P-QM]_H} \longrightarrow \mathcal{C}^{u.e.v,r.g.}_{[P-QI]_{r.s.}} & & \textit{is } \text{full and faithful.}
\end{array}$$

Moreover, Hyp *and* ∂_∞ *are mutually inverse functors between* $\mathcal{C}^{u.e.,v,r.g.}_{[P-QI]_{r.s.}}$ *and* $\mathcal{D}^{u.p.}_{[P-QM]_H}$. □

Remark 122. *Note that the functor* Hyp *has a "built-in" faithfulness, which corresponds to the fact that there are intuitively "more" maps between two hyperbolic spaces than there are maps between their boundaries. This is why* ∂_∞ *can be non-faithful and* Hyp *need not be full.*

What is more surprising is that ∂_∞ *is not necessarily full either. This can be seen for example by taking two mutually non-quasi-isometric unbounded subsets of* $\mathbb{R}_{\geq 0}$*. Then the identity map between their (one-point) boundaries at infinity cannot come from a quasi-isometric map between the spaces. In particular,* ∂_∞ *(as defined in the first row in Prop. 121 above) is not full.*

Chapter 8

Asymptotic and Assouad-Nagata Dimensions

In this Chapter we (implicitly) apply the extension theorems to prove a lower bound of the linear asymptotic dimension of a visual roughly geodesic Gromov hyperbolic space in terms of the Assouad-Nagata dimension of its boundary at infinity.

This can be combined with estimates in the other direction such as Cor. 12.1.11 in [BS07], cf. Thm. 130 below, to yield a very precise correspondence between the asymptotic dimension of a visual roughly geodesic Gromov hyperbolic space and the Assouad-Nagata dimension of its boundary at infinity. More precisely, the following holds.

Proposition 123. *Let X a visual roughly geodesic Gromov hyperbolic metric space. Let $\partial_\infty X$ be its boundary at infinity be equipped with any Bourdon quasi-metric. Then*

$$ANdim\,\partial_\infty X = \ell\text{-}asdim\,X \qquad \text{or} \qquad ANdim\,\partial_\infty X = \ell\text{-}asdim\,X - 1.$$

In [BL05], Buyalo-Lebedeva proved that $\ell\text{-asdim}\,X = ANdim\partial_\infty X + 1$ for any cobounded visual geodesic hyperbolic space X where the boundary $\partial_\infty X$ is equipped with a bounded visual metric. In fact, in this case the asymptotic and the linear asymptotic dimensions of X coincide, and so do the topological, linearly controlled metric and the Assouad-Nagata dimensions of the boundary $\partial_\infty X$.

We recall the relevant definitions. We refer to [BS07] Ch. 9 for a more detailed introduction to dimension theory.

Definition 124. *Let $\mathcal{U}$ be an open covering of a metric space X.*

The mesh size *of $\mathcal{U}$,* mesh($\mathcal{U}$)*, is defined as* $\text{mesh}(\mathcal{U}) := \sup\{\text{diam}\,U : U \in \mathcal{U}\}$.

The multiplicity *of $\mathcal{U}$, $m(\mathcal{U})$ is the maximal number of members of $\mathcal{U}$ with nonempty intersection.*

Given $x \in X$, we let

$$L(\mathcal{U}, x) = \sup\{\text{dist}(x, U^c) : U \in \mathcal{U}\}$$

be the Lebesgue number *of $\mathcal{U}$ at x and $L(\mathcal{U}) = \inf_{x\in X} L(\mathcal{U}, x)$ the Lebesgue number of $\mathcal{U}$.*

Definition 125 (*ANdim* for metric spaces)**.** *The* Assouad-Nagata dimension *of a metric space X, $ANdim\, X$, is the minimal integer n with the following property: there exists $\delta \in (0,1)$ such that for every positive τ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le n+1$, $\mathrm{mesh}(\mathcal{U}) \le \tau$ and $L(\mathcal{U}) \ge \delta\tau$.*

Definition 126. *The* linearly controlled metric dimension *of a metric space X, ℓ-$dim\, X$, is the minimal integer n with the following property: there exists $\delta \in (0,1)$ such that for every sufficiently small $r > 0$ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le n+1$, $\mathrm{mesh}(\mathcal{U}) \le r$ and $L(\mathcal{U}) \ge \delta r$.*

Definition 127. *The* linearly controlled asymptotic dimension *of a metric space X, ℓ-$asdim\, X$, is the minimal integer n with the following property: there exists $\delta \in (0,1)$ such that for every sufficiently large $R > 1$ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le n+1$, $\mathrm{mesh}(\mathcal{U}) \le R$ and $L(\mathcal{U}) \ge \delta R$.*

Remark 128. *The linear asymptotic dimension is invariant under quasi-isometries, cf. [BS07] 9.1.2. The Assouad-Nagata dimension is invariant under quasimoebius maps, cf. [LS05], [Xie08].*

It is also well-known that $ANdim\, X = \max\{\ell\text{-}dim\, X, \ell\text{-}asdim\, X\}$. In particular, for any bounded *metric space X, $ANdim\, X = \ell$-$dim\, X$.*

Since the Assouad-Nagata dimension is known to be invariant under quasimoebius maps, the following generalization is very natural.

Definition 129 (*ANdim* for quasimetric spaces)**.** *Let (Z, ρ) a quasimetric space. The Assouad-Nagata dimension of Z, $ANdim\, Z$ is defined to be the Assouad-Nagata dimension of (Z, d), where d is any bounded metric on Z that is bilipschitz-quasimoebius equivalent to ρ^α for some $\alpha > 0$.*

The Assouad-Nagata dimension is then a quasimoebius-invariant dimension in the category of quasimetric spaces.

The following dimension bounds are known.

Theorem 130 ([BS07] Cor. 12.1.11)**.** *Let X a visual Gromov hyperbolic space, then*

$$asdim\, X \le \ell\text{-}asdim\, X \le \ \ell\text{-}dim\, \partial_\infty X + 1.$$

If X is in addition proper, geodesic and cobounded, then equality holds everywhere.

For our bound in the opposite direction, we produce a cover of the boundary out of the cover of the space. This is achieved via the intuitive notion of the *shadow* that a subset $U \subset X$ of a hyperbolic space casts in the boundary $\partial_\infty X$. This notion has an extremely simple realization in case X is a hyperbolic approximation and U a subset of the vertex set. Namely the shadow of a vertex is nothing but its associated ball in $\partial_\infty X$. Then one can define the shadow of a set of vertices as the union of the shadows of its elements. However, we define it slightly different for technical reasons which will become clear in the proof of Lemma 133.

In the following definition we abuse notation and denote by $U \subset \mathcal{V}$ a subset of the set $\bigcup_l V_l$.

Definition 131. *Let X a hyperbolic approximation of some complete K-quasimetric space (Z, ρ) and let $v \in V_l$ a vertex of some level l. The* shadow of v*, $sh(v)$, is the set $B_{Kr^l}(\pi(v)) \subset Z$.*

If $U \subset \mathcal{V}$ is a set of vertices of X with $|U| \geq 2$, the shadow of U*, $sh(U)$, is the union of all shadows of vertices of U which are* not *horizontally connected to any vertex in U^c.*

Remark 132. *There is a slight ambiguity in this definition since a K-quasimetric space is also a K'-quasimetric space for every $K' > K$, and maybe even for some $K' < K$. This could be remedied by taking an* optimal*, i.e. least possible, K such that (Z, ρ) is still a K-quasimetric space. However, in our proof of Prop. 123 the exact K will not matter anyway, so we are not concerned by this ambiguity.*

The reason why we do not define $sh(U)$ as just the union of *all* shadows of vertices is because we want the following.

Lemma 133. *Suppose X is a hyperbolic approximation of some quasimetric space (Z, d). Let U, U' two disjoint subsets of the vertex set V_l, for some level l. Then $sh(U) \cap sh(U') = \emptyset$.*

Proof. Let $z \in Z$ and suppose $z \in sh(U)$. Then $z \in B(v)$ for some $v \in U$ and v is not horizontally connected to U^c, in particular not connected to any vertex in U'. But this means z cannot be in any $B(w)$, $w \in U'$, as otherwise v and w would be connected. In particular, $z \notin sh(U')$. □

Proposition 134. *Let X be a truncated hyperbolic approximation of a metric space with $\ell - asdim\, X = n$ and let $\lambda \in (0,1)$ and $R > 1$ be such that X can be covered with a system $\mathcal{U}'$ of sets with mesh $\leq R$ but Lebesgue number $\geq \lambda R > 1$ and multiplicity $n+1$. Let $\mu := a^{-(\frac{1-\lambda}{2}R)-1} \in (0,1)$.*

Then for any d small enough $\partial_\infty^{a;o} X$ can be covered with a system of sets with mesh $\leq d$, Lebesgue number $\geq \mu d$ and multiplicity $\leq n+1$. That is to say, ℓ-$dim\, \partial_\infty X = ANdim\, \partial_\infty X \leq n$.

Proof. First we explain how to get the cover of $\partial_\infty X$. Set $r_0 := \left\lceil \frac{\log(1/d)}{\log a} + R/2 \right\rceil$. Take the set V of all vertices v such that $|ov| = r_0$, where o is the root of the truncated approximation. Thus v is nothing but the vertex set V_{r_0} of level r_0. Then take $\mathcal{U} := \mathcal{U}' \cap V$.

It is clear that the system of shadows $sh(\mathcal{U})$ covers $\partial_\infty X$, because for each vertex $v \in V_{r_0}$ there is a set $U \in \mathcal{U}$ such that v is in U and not connected to U^c. In other words, $sh(v)$ will be contained in $sh(U)$. Since the balls $B(v), v \in V_{r_0}$ cover $\partial_\infty X$, $sh(\mathcal{U})$ is a cover of $\partial_\infty X$.

Also, the multiplicity of the covering $sh(\mathcal{U})$ is $\leq n+1$ because disjoint sets in $\mathcal{U}$ produce disjoint shadows by Lemma 133.

As for the mesh size, since for any $x, y \in U$, $U \in \mathcal{U}$ arbitrary, we have $|ox| = |oy| = r_0$ and $|xy| \leq R$. The choice of r_0 yields that $diam_{\partial_\infty^{a;o} X}(sh(U)) \leq d$, so the mesh size of the boundary cover is $\leq d$ as required.

Must have control over the Lebesgue number. So let $\xi \in \partial_\infty X$ arbitrary. Since X is a hyperbolic approximation we can take a geodesic ray $o\xi$ and on this ray there is a unique vertex x with $|ox| = r_0$. Pick $U_\xi \in \mathcal{U}$ such that x is at least λR from U_ξ^c. Then for any $\xi' \notin sh(U_\xi)$ we calculate

$$(x|x')_o = 1/2(|xo| + |x'o| - |xx'|) = r_0 - 1/2|xx'| \leq r_0 - \frac{\lambda}{2}R + 1,$$

where x' is the unique point of level r_0 on a ray $o\xi'$ and the $+1$ comes from the fact that x' might be a point in U_ξ connected to U_ξ^c.

Thus

$$a^{-(\xi|\xi')} = a^{-(x|x')_o} \geq a^{-r_0 + \frac{\lambda}{2}R - 1}$$

so

$$a^{-(\xi|\xi')_o} \geq da^{-R/2 + \lambda R/2 - 1} = da^{\frac{\lambda - 1}{2}R - 1} = \mu d.$$

This proves the proposition. □

Proof of Prop. 123. Since the Assouad-Nagata dimension is invariant under quasimoebius maps and the asymptotic dimension under quasi-isometries, it is enough to show the equalities for the case where $\partial_\infty X$ is equipped with a visual metric bilipschitz equivalent to a quasimetric of the form $a^{-(\cdot|\cdot)_o}$, and X is a truncated hyperbolic approximation of $\partial_\infty X$.

But then by Prop. 134, $\ell\text{-dim}\, \partial_\infty X \leq \ell\text{-asdim}\, X$. On the other hand by Thm. 130, $\ell\text{-asdim}\, X \leq \ell\text{-dim}\, \partial_\infty X + 1$. Together with Rem. 128 these imply the claim. □

For most reasonable spaces one would actually expect $ANdim\, X = \ell\text{-asdim}\, X - 1$, see for example [BL05] Thm. 6.4. However, for the space $X = \{2^n \,|\, n \in \mathbb{N}\}$ we have $\ell\text{-asdim}\, X = ANdim\, \partial_\infty X = 0$. This X is of course not geodesic. It is unclear if there are more interesting examples of $\ell\text{-asdim}\, X = ANdim\, \partial_\infty X$.

Chapter 9

Outlook and Further Questions

Here we collect some open problems related to various sections.

9.1 Boundary at Infinity

Several seemingly elementary questions regarding visual metrics remain open. For example it is well-known that in the case of CAT(-1)-spaces, $e^{-(\cdot|\cdot)_o}$ is in fact a metric, i.e. satisfies the triangle inequality. We have also seen examples where $e^{-(\cdot|\cdot)_o}$ is not bilipschitz to a metric. However, there seems to be no example in the literature of a space X such that $e^{-(\cdot|\cdot)_o}$ *is* bilipschitz to a metric, but not itself a metric.

Question 135. *Suppose X is a visual Gromov hyperbolic metric space and $a > 1$ such that $a^{-(\cdot|\cdot)_o}$ is bilipschitz to a metric on $\partial_\infty X$. Is it true that $a^{-(\cdot|\cdot)_o}$ automatically satisfies the triangle inequality? If not, what is an example of such an X? What kind of properties of a space X influence whether or not $a^{-(\cdot|\cdot)_o}$ satisfies the triangle inequality?*

Another question is related to the asymptotic curvature and the chain construction. It is known that the chain construction applied to a 2-quasimetric yields a metric bilipschitz equivalent to the original quasimetric. It is also known that for every $\epsilon > 0$ there are $(2+\epsilon)$-quasimetric spaces where the chain construction does not produce a metric, but only a pseudo-metric ([Sch06]). Bonk and Foertsch prove in addition ([BF06] Lemma 4.3) that if a quasimetric ρ satisfies

$$\rho(z, z') \leq Cn^\alpha \max_i \rho(z_{i-1}, z_i),$$

for some fixed constants $C > 0$, $0 < \alpha < 1$ and all chains from z to z', then the chain construction also yields a metric bilipschitz equivalent to ρ.

Question 136. *Suppose (Z, ρ) is a quasimetric space where ρ satisfies a* linear chain condition*, i.e. there exists a constant C such that*

$$\rho(z, z') \leq Cn \max_i \rho(z_{i-1}, z_i), \tag{9.1}$$

for all chains $z = z_0, z_1, \ldots, z_n = z'$.

Is it true that there is a K such that $\rho(z, z') \leq K \sum \rho(z_{i-1}, z_i)$ for all $z, z' \in Z$ and chains $z_0, \ldots, z_n$ between them?

Intuition would suggest that the answer is probably "no", but having failed repeatedly to cook up an example, the author is now inclined to believe that it may just be true, i.e. that a quasimetric which satisfies a linear chain condition (9.1) is indeed bilipschitz equivalent to a metric.

9.2 Hyperbolic Approximation

Is there a direct construction of a graph Hyp_r as described in §5.1 for any $r \in (0, 1)$ such that the requirements one has on its boundary are still satisfied? This would allow to drop the unaesthetic scaling procedure we applied in §5.2

9.3 Extension Theorems

Besides roughly isometric and power quasi-isometric maps there are other classes of maps between Gromov hyperbolic spaces that canonically induce maps between their boundaries. Is it possible to recover them from their induced boundary maps? For example, consider the following class of maps which lie in between quasi-isometric and power quasi-isometric maps (recall Ex. 28 3, Rem. 29).

Definition 137. *Call a map $F : X \to Y$ between Gromov hyperbolic spaces* weakly power quasi-isometric*, or wP-QI for short, if there are constants c, d such that for every quadruple $Q \subset X$ the following is satisfied:*

$$\frac{1}{c}\mathrm{wcd}(Q) - d \leq \mathrm{wcd}(Q') \leq c\,\mathrm{wcd}(Q) + d.$$

Here, the *weak cross-difference*, wcd, of a quadruple is defined as follows.

Definition 138. *For a quadruple $Q = (x, y, z, w) \subset X$ in a metric space, the weak cross-difference of Q,* $\mathrm{wcd}(Q)$*, is the number*

$$\mathrm{wcd}(Q) := \max_{a, a' \in A(Q)} (a - a'),$$

where $A(Q)$ is the cross-difference triple of Q as defined in (22)*, $A(Q) = \{|xz| + |yw|, |xy| + |zw|, |xw| + |yz|\}$.*

Remark 139. *Every quasi-isometric map is wP-QI and every wP-QI map is P-QI. The reverse inclusions are generally not true, except when the spaces are geodesic Gromov hyperbolic spaces, Thm. 31.*

Power quasi-isometric maps in our sense are called strongly power quasi-isometric *in [BS07], while weakly power quasi-isometric maps in our sense are called* power quasi-isometric.

Every weakly power quasi-isometric map F between Gromov hyperbolic spaces induces a boundary map $\partial_\infty F$ which has certain "weak" power quasi-moebius properties (see [BS07] Prop. 5.2.10 for details). An example of such a wP-QI map is the map F of Ex. 28 3. It induces a boundary map

$\partial_\infty F : \partial_\infty X \to \partial_\infty Y$ that is not P-QM in the sense of our Def. 46. Nevertheless, it satisfies the slightly weaker notion of power quasimoebius in the sense of [BS07] Def. 5.2.7. Let us call such maps *weakly* power quasimoebius

Question 140. *Given a weakly power quasimoebius map $f : Z \to Z'$ between quasimetric spaces. Is there a way to produce Gromov hyperbolic spaces X, Y and a weakly power quasi-isometric $F : X \to Y$ such that $\partial_\infty F = f$?*

9.4 P-QM and P-QI Invariants

Every property of quasimetric spaces which is invariant under P-QM maps corresponds to a property of visual geodesic Gromov hyperbolic spaces which is invariant under P-QI maps. For example, the property of uniform perfection of quasimetric spaces corresponds to the property to be uniformly equilateral among visual geodesic Gromov hyperbolic spaces (cf. [Mey09]). Similarly, in Ch. 8we have a correspondence between the Assouad-Nagata dimension of a quasimetric space and the linearly controlled asymptotic dimension of a Gromov hyperbolic space.

Consider other, maybe as of yet unknown, P-QM-invariant properties of quasimetric spaces and interpret their corresponding P-QI-invariant property of visual geodesic Gromov hyperbolic spaces, and vice versa.

9.5 Embeddings into CAT(-1) Spaces

One of the main results of [BS00] states that a Gromov hyperbolic space X that has *bounded growth at some scale* (see [BS00] for details) can be "almost" roughly isometrically embedded into standard hyperbolic space $\mathbb{H}^n$, for some $n \in \mathbb{N}$. Namely if X satisfies the mentioned conditions and $\lambda \in (0, 1)$, then there exists an $n \in \mathbb{N}$ such that the scaled space λX embeds roughly isometrically into $\mathbb{H}^n$. The proof is a rather straightforward (a slight detour is necessary for spaces which are not visual) application of the miraculous Assouad Embedding Theorem (see [Ass83], or [BS07] Thm. 8.1.1), which states that if (Z, d) is a doubling metric space, then for every $p \in (0, 1)$ there exists an $N \in \mathbb{N}$ such that (Z, d^p) embeds bilipschitz into $\mathbb{R}^N$.

Using a warped product approach to hyperbolic cone constructions and the curvature bounds for warped products of certain classes of metric spaces due to Alexander and Bishop (see [AB04]) one can show that any visual hyperbolic space whose boundary embeds (in a bilipschitz manner) for example into a CAT(0) space will embed roughly isometrically into a CAT(-1) space. Foertsch and Schroeder use this in [FS06] to show that for any hyperbolic space (except possibly some pathological *treelike* cases, see the given reference) whose boundary has finite Assouad-Nagata dimension there exist $\lambda \in (0, 1), N \in \mathbb{N}$ such that λX embeds roughly isometrically into a CAT(-1) space. The proof is based on a result by Lang and Schlichenmaier (see [LS05]), which says that a metric space with finite Assouad-Nagata dimension admits a Hölder embedding into a product of trees, which is a CAT(0) space.

The author proposes the following

Conjecture 141. *Let (Z, d) be any metric space. Then $(Z, d^{1/2})$ admits a bilipschitz embedding into a CAT(0) space.*

The motivation for this conjecture comes from the fact that the square root of a metric space is a metric space that satisfies both the Ptolemy and the Quadrilateral Inequality. Background on these inequalities can be found for example in [FS06] and [BN08] respectively. In essence, they both hint to the possibility that a metric space satisfying them might be a subset of a CAT(0) space. While there are easy examples of geodesic Ptolemy spaces that are not CAT(0), these are based on the fact that a Ptolemy space can have two distinct midpoints for a given pair of points. However, the square root of a metric will never have midpoints. The idea to prove the conjecture would be to somehow "fill in" the space $(X, d^{1/2})$ by inductively inducing midpoints such that the Quadruple Inequality remains satisfied. If this procedure could be iterated and the original metric not changed (or at least only bilipschitz distorted), we would get a geodesic limit space which satisfies the Quadrilateral Inequality, and such a space is CAT(0) by [BN08].

Naturally, if this conjecture were true, it would entail that pretty much every visual Gromov hyperbolic space has a roughly similar embedding into a CAT(-1) space, which would be rather surprising.

Appendix A

Invariance of the Hyperbolic Approximation

We justify here the claim that the construction of the hyperbolic approximation described in Ch. 5 is well-defined up to rough isometry, i.e. independent of the choice of vertex system and choice of the quasimetric constant K used for (Z, ρ).

Lemma 142 (Independence of the vertex system)**.** *Two hyperbolic approximations with the same r and K of the same quasimetric space but different vertex sysmtems are roughly isometric.*

Proof. Define the rough isometry $F : X \to X'$ level-wise. For each level k map a vertex $v \in V_k$ to a vertex $F(v) \in V'_k$ such that $B(F(v))$ contains v. $\square$

For the independence of K, the most elegant proof is to use the bilipschitz extension theorem. However, we should also have a version of this for extended boundaries.

Theorem 143 (Compare Thm. 75)**.** *Suppose X, X' are visual geodesic Gromov hyperbolic spaces and b, b' two Busemann functions on $\partial_\infty X, \partial_\infty X'$ respectively. Then for any bilipschitz map $f : \partial_\infty^{a,b} X \to \partial_\infty^{a,b'} X'$ there exists a roughly isometric map $F : X \to X'$ with $\partial_\infty F = f$.*

Proof. It is enough to prove it in the case where $b = b_{\omega,o}, b = b_{\omega',o'}$ for some $\omega \in \partial_\infty X$, $o \in X$, $\omega' \in \partial_\infty X'$, $o' \in X'$ (recall Def. 41) and assuming that for every $\xi \in \partial_\infty X$ and every $\xi' \in \partial_\infty X'$ there exists a bi-infinite geodesic $\omega\xi$ and $\omega'\xi'$ respectively. Moreover, assume for every $x \in X$ there is a bi-infinite geodesic that passes through x and has one end in ω. Note that all these requirements are tailored for the case when X, X' are the graphs of hyperbolic approximations.

Define F as follows. For $x \in X$ choose a bi-infinite geodesic through x with one end in ω and call the other end $\xi \in \partial_\infty X$. Then in X', consider a bi-infinite geodesic $\omega' f(\xi)$, and define $F(x)$ to be a point on this geodesic with $b'(F(x)) = b(x)$.

Since f is bilipschitz, we have $(f(\xi)|f(\eta))_{b'} \doteq (\xi|\eta)_b$ for all $\xi, \eta \in \partial_\infty X$. Now it is straightforward to check that $(\xi|\eta)_b \doteq (x|y)_b$ (when x, y do not lie on a common ray this is immediate. If they do happen to lie on one common ray, it is also not difficult to see). Likewise, $(f(\xi)|f(\eta))_{b'} \doteq (F(x)|F(y))_{b'}$ by the definition of F. In other words, we have

$$(x|y)_b \doteq (F(x)|F(y))_{b'},$$

and that implies $|xy| \doteq |F(x)F(y)|$ because $b(x) = b'(F(x)), b(y) = b'(F(y))$. $\square$

Remark 144. *Note that in the proof of the extension theorem for inversions Thm. 92, we used independence of the approximation on K, so we cannot combine that theorem with the bilipschitz extension theorem for bounded spaces to get Thm. 143 for free.*

Lemma 145 (Independence of K)**.** *Let (Z,ρ) a K-quasimetric space, $K' \geq K$, $r \in (0,1)$ and $\mathrm{Hyp}\,_r^K(Z), \mathrm{Hyp}\,_r^{K'}(Z)$ two hyperbolic approximations, one assuming ρ to be a K-quasimetric, and the other assuming ρ to be merely a K'-quasimetric. Then $\mathrm{Hyp}\,_r^K(Z) \doteq \mathrm{Hyp}\,_r^{K'}(Z)$.*

Proof. It is enough to show it for $r = 1/K^3$. For illustrative purposes (and since it is the only case we needed in this thesis) we show it for $K' = K^2$. Let $a = 1/r$. Then $\mathrm{Hyp}\,_r^{K'}(Z,\rho)$ is by definition the graph of $\mathrm{Hyp}\,_r(Z,\rho^{1/2})$ scaled by a factor of two (i.e. each edge has length 2). But the boundary of $\mathrm{Hyp}\,_r(Z,\rho^{1/2})$ equipped with $a^{-(\cdot|\cdot)}$ is bilipschitz equivalent to $(Z,\rho^{1/2})$. Consequently, the boundary of $\mathrm{Hyp}\,_r^{K'}(Z,\rho)$ equipped with $a^{-(\cdot|\cdot)}$ is bilipschitz equivalent to (Z,ρ). The same is also true for $\mathrm{Hyp}\,_r^K(Z,\rho)$, hence the claim follows from Thm. 75 in the bounded case and from Thm. 143 in the extended case. $\square$

Appendix B

Between Bounded Spaces P-QM is P-QS and BL-QM is BL

We give here a proof of the fact that a power quasimoebius map between bounded spaces is power quasisymmetric. Likewise, a bilipschitz quasimoebius map between bounded spaces is in fact bilipschitz. These results are essentially due to Väisälä, who proved in [Väi85] Thm. 3.12 (see also [Väi99]) that a quasimoebius map between bounded metric spaces is quasisymmetric. However, it is not immediately obvious from his proof that the quasiysmmetric control function can be taken of the same type as the quasimoebius control function, i.e. still of power type and linear respectively.

Theorem 146 (Compare [Väi99] 6.29(2))**.** *Suppose X and Y are bounded K-quasimetric spaces and $f : X \to Y$ is P-QM (BL-QM). Suppose also that $\lambda > 0, z_1,\, z_2,\, z_3 \in X$ are such that*

$$|z_i z_j| \geq d(X)/\lambda,\ |f(z_i)f(z_j)| \geq d(Y)/\lambda$$

for $i \neq j$. Then there is a homeomorphism $\mu : [0,\infty) \to [0,\infty)$ of power-type (linear) such that

$$\frac{|f(x)f(y)|}{d(Y)} \leq \mu\left(\frac{|xy|}{d(X)}\right),$$

and moreover, f is P-QS (BL-QS, and hence bilipschitz by Thm. 106).

Proof. We first prove the assertion on μ. Suppose f is θ-QM, with θ of power type (linear). We may assume that f is a homeomorphism and that f^{-1} is also θ-QM. We can normalize so that $d(X) = d(Y) = \lambda$ by scaling X by $\lambda/d(X)$ and analogously for Y.

Let $x, y \in X$. Let $r = |xy|,\ r' = |x'y'|$, where the prime denotes images under f. Show $r' \leq \mu(r)$ with μ as claimed. Consider three cases:

Case 1: $r \geq 1/K$. then $r' \leq \lambda \leq 4\lambda r$.

Case 2: $|xz_1| \geq 1/K$. There is $z \in \{z_2, z_3\}$ with $|yz| \geq 1/K$. For $Q = (x, z_1, y, z)$ we then have $cr(Q) \leq K^2\lambda r$ and $cr(Q') \geq r'/\lambda^2$, whence $r' \leq \lambda^2\theta(K^2\lambda r)$.

Case 3: $|xz_1| < 1/K$, $r < 1/K^2$. Then $|xz_j| \geq 1/K$ and $|yz_j| \geq 1/K^2$ for $j = 2, 3$. For $Q = (x, z_2, y, z_3)$ we thus have $cr(Q) \leq K^3 \lambda r$ and $cr(Q') \geq r'/\lambda^2$. Consequently, $r' \leq \lambda^2 \theta(K^3 \lambda r)$.

In all three cases we obtain an upper bound for r' in terms of $\lambda\cdot$ a power-type (linear) function of r. It is easy to find a power-type (linear) function $\tilde{\mu}$ which is bigger than all three. Then $r' \leq \lambda \tilde{\mu}(r)$, showing the claim by setting $\mu(t) := \tilde{\mu}(\lambda t)$, which is still of power type (linear).

Prove now that f is P-QS (BL-QS). Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) := \lambda \mu(t/\lambda)$. Then ψ is of power type (linear) and, by the part above applied to f and f^{-1} (recall f^{-1} has the same control function as f)

$$\psi^{-1}(|xy|) \leq |x'y'| \leq \psi(|xy|) \; \forall x, y \in X.$$

Let $T = (x, a, b)$ a triple in X. Recall the notation for the standard ratio $sr(T) := |xz|/|xy|$ of a triple. By Thm. 147, it is enough to show $sr(T') \leq \eta(sr(T))$, with η of the form "power + constant" (affine). Since $|z_2 z_3| \geq 1$, we may assume that $|az_2| \geq 1/K$. Consider again three cases.

Case 1: $|ax| \geq 1/K^2$. Then by definition, $|bx| \geq 1/(K^2 sr(T))$, thus $|b'x'| \geq \psi^{-1}(1/(K^2 sr(T)))$, which gives the desired bound with $\eta(t) = \lambda/\psi^{-1}(1/K^2 t)$. Note that since ψ is of power type (linear), so is η.

Case 2: $|bz_2| \geq 1/K^3$. Then $|b'z_2'| \geq \psi^{-1}(1/K^3)$. With $Q = (x, b, a, z_2)$, we have $cr(Q) \leq K \lambda sr(T)$, and since $cr(Q') \geq \psi^{-1}(1/K^3) sr(T')/\lambda$, we get the desired estimate with

$$\eta(t) = \frac{\lambda \theta(2\lambda t)}{\psi^{-1}(1/K^3)}.$$

Case 3: $|ax| < 1/K^2$ and $|bz_2| < 1/K^3$. Then $|bx| \geq 1/K^3$. Thus $|b'x'| \geq \psi^{-1}(1/K^3)$, whence we get the desired estimate with the constant function $\eta(t) = \lambda/\psi^{-1}(1/K^3)$.

The first two cases resulted in estimates where η was of power-type (linear), while the last case gives a constant. All in all we can thus produce a function η of "power-type plus constant" (affine) such that $sr(T') \leq \eta(sr(T))$. Since the same estimates can be obtained for f^{-1}, Theorem 147 now implies the claim. $\square$

Theorem 147. *Suppose $f : X \to Y$ is an embedding, $\eta : [0, \infty) \to [0, \infty)$ of the form $\eta(t) = C \max\{t^p, t^{1/p}\} + D$ ($\eta(t) = \lambda t + D$), such that*

1. *$sr(T') \leq \eta(sr(T))$ for each triple $T \subset X$,*
2. *$sr(f^{-1}(T')) \leq \eta(sr(T'))$ for each triple $T' \subset Y$.*

Then f is PQ-S (BL-QS).

Proof. Let $T = (x, a, b)$ triple in X. We have to show that $sr(T') \leq \eta_1(t)$ with η_1 of power-type (linear). Suppose $\eta(0) = r_0$. Set $t_0 := 1/r_0$. Define the homeomorphism $\eta_0 : (0, t_0) \to (0, \infty)$ by $\eta_0(t) := 1/\eta^{-1}(t^{-1})$, i.e. in particular $\eta_0(0) = 0$, "$\eta(t_0) = \infty''$.

Now if T is a triple with $r(T) < t_0$, consider the triple $\tilde{T}' = (x', b', a')$, i.e. b and a reversed. Then applying 2) to $\tilde{T}$ gives

$$\frac{1}{sr(T)} = sr(\tilde{T}) \leq \eta(sr(\tilde{T}') = \eta(1/sr(T')),$$

i.e.

$$sr(T) \geq \frac{1}{\eta(1/sr(T'))},$$

hence

$$\eta_0(sr(T)) \geq \eta_0(\frac{1}{\eta(1/sr(T'))}) = \frac{1}{\eta^{-1}(\eta(1/sr(T')))} = sr(T').$$

Thus we have $sr(T') \leq \eta_1(sr(T))$ for all triples with η_1 the minimum of η, and η_0. Now η_0 is not of power type. However, there is an $s_0 < t_0$ where η_0 starts to become bigger than η. Consider η^{-1} restricted to $[1/s_0, \infty) \subset [r_0, \infty)$. It is easy to find a power type (linear) function σ such that σ^{-1} is smaller than $\eta^{-1}|_{[1/s_0,\infty)}$, cf. Fig. below. Thererfore we will have $1/\sigma^{-1}(1/t) \geq \eta^{-1}(1/t) = \eta_0(t)$ for $t \leq s_0$. In particular, $sr(T') \leq \eta_2(t) := 1/\sigma^{-1}(1/t)\ \forall t \leq s_0$, with η_2 a power-type (linear) function.

Set $\eta_3 := \min\{\eta_0, \eta_2\}$ (by abuse of notation, the domain of η_3 is the union of the domains and it is the minimum of the two on the overlap). It is now easy to find a single power-type (linear) function η_1 which is everywhere greater than η_3. Then $sr(T') \leq \eta_1(sr(T))$ for any triple $T \subset X$ with $r(T) < t_0$.

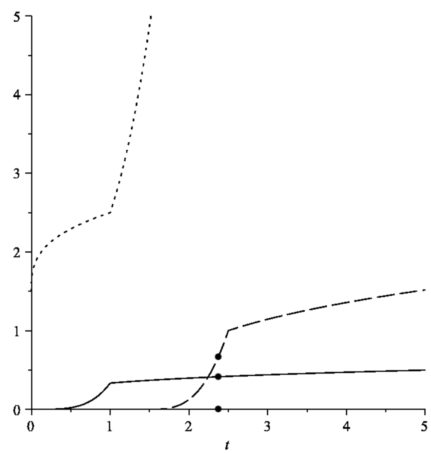

Figure B.1: Situation as in the proof of Thm. 147, with $s_0^{-1} \approx 2.4$.

□

Bibliography

[AB04] S. B. Alexander and R. L. Bishop. Curvature bounds for warped products of metric spaces. *Geom. Funct. Anal.*, 14(6):1143–1181, 2004.

[Ass83] Patrice Assouad. Plongements lipschitziens dans $\mathbf{R}^n$. *Bull. Soc. Math. France*, 111(4):429–448, 1983.

[BF06] M. Bonk and T. Foertsch. Asymptotic upper curvature bounds in coarse geometry. *Math. Z.*, 253:753–785, 2006.

[BH99] M. Bridson and A. Haefliger. *Metric Spaces of Non-Positive Curvature.* Springer, 1999.

[BL05] S. Buyalo and N. Lebedeva. Dimensions of locally and asymptotically self-similar spaces. arXiv:math.GT/0509433, 2005.

[BM91] Mladen Bestvina and Geoffrey Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, 4(3):469–481, 1991.

[BN08] I. D. Berg and I. G. Nikolaev. Quasilinearization and curvature of Aleksandrov spaces. *Geom. Dedicata*, 133:195–218, 2008.

[Bow98a] B. H. Bowditch. Boundaries of strongly accessible hyperbolic groups. In *The Epstein birthday schrift*, volume 1 of *Geom. Topol. Monogr.*, pages 51–97 (electronic). Geom. Topol. Publ., Coventry, 1998.

[Bow98b] Brian H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.

[BP03] M. Bourdon and H. Pajot. Cohomologie l_p et espaces de Besov. *J. reine angew. Math.*, 558:85–108, 2003.

[BS00] M. Bonk and O. Schramm. Embeddings of Gromov hyperbolic spaces. *Geom. Funct. Anal.*, 10:266–306, 2000.

[BS07] S. Buyalo and V. Schroeder. *Elements of Asymptotic Geometry.* EMS Monographs in Mathematics. European Mathematical Society, 2007.

[Ele97] G. Elek. The l_p-cohomology and the conformal dimension of hyperbolic cones. *Geom. Dedicata*, 68:263–279, 1997.

[Fri37] A.H. Frink. Distance functions and the metrization problem. *Bull. Amer. Math. Soc.*, 43:133–142, 1937.

[FS06] T. Foertsch and V. Schroeder. Hyperbolicity, CAT(-1)-spaces and the Ptolemy Inequality. *ArXiv Mathematics e-prints*, May 2006.

[GdlH90] Étienne Ghys and Pierre de la Harpe. Espaces métriques hyperboliques. In *Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988)*, volume 83 of *Progr. Math.*, pages 27–45. Birkhäuser Boston, Boston, MA, 1990.

[Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, Math. Sci. Res. Inst. Publ. 8, pages 75–263, New York, 1987. Springer Verlag.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Hei01] J. Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[Jor10] J. Jordi. Interplay between interior and boundary geometry in Gromov hyperbolic spaces. *Geometriae Dedicata*, pages 1–26, 2010. 10.1007/s10711-010-9472-0.

[LS05] U. Lang and T. Schlichenmaier. Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. *Int. Math. Res. Not.*, 58:3625–3655, 2005.

[Mar08] Á. Martínez-Pérez. Quasi-isometries between visual hyperbolic spaces. *ArXiv e-prints*, October 2008.

[Mey09] J. Meyer. *Uniformly perfect boundaries of Gromov hyperbolic spaces.* PhD thesis, University of Zurich, 2009.

[Mor21] H.M. Morse. Recurrent geodesics on a surface of negative curvature. *Trans. Amer. Math. Soc.*, 22:84–100, 1921.

[Mor24] H.M. Morse. A fundamental class of geodesics on any closed surface of genus greater than one. *Trans. Amer. Math. Soc.*, 26:25–60, 1924.

[Pau96] F. Paulin. Un groupe hyperbolique est déterminé par son bord. *J. London Math. Soc.*, 54(2):50–74, 1996.

[Poi85] Henri Poincaré. *Papers on Fuchsian functions.* Springer-Verlag, New York, 1985. Translated from the French and with an introduction by John Stillwell.

[Sch06] V. Schroeder. Quasi-metric and metric spaces. *Conform. Geom. Dyn.*, 10:355–360, 2006.

[Sch09] V. Schroeder. An introduction to asymptotic geometry. To appear in IRMA lectures in mathematics and theoretical physics, 2009.

[TV80] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5:97–114, 1980.

[TV99] D.A. Trotsenko and J. Väisälä. Upper sets and quasisymmetric maps. *Ann. Acad. Sci. Fenn. Fenn.*, 24:465–488, 1999.

[Väi99] J. Väisälä. The free quasiworld. *Banach Center Publ.*, 48:55–118, 1999.

[Väi85] J. Väisälä. Quasimöbius maps. *J. Analyse Math.*, 44:218–234, 1984/85.

[Xie08] X. Xie. Nagata dimension and quasi-möbius maps. *Conf. Geom. Dyn.*, 12:1–9, 2008.